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# Variational Functionals for the Characterization of BV and Sobolev Spaces

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**Abstract.** This paper examines the characterization of bounded variation ($BV$) and Sobolev functions by using some non-local functionals. We analyze their pointwise convergence and establish a connection with the Sobolev norm and the total variation measure. By investigating a wider class of non local functionals, we provide a deeper understanding of how these approximations capture the local properties of $BV$ and Sobolev spaces, thereby reinforcing their applicability in the Euclidean setting.

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## 1 Introduction

Sobolev spaces, denoted by $W^{1,p}(\mathbb{R}^n), 1 \leq p < \infty$, are fundamental in the study of functional analysis and partial differential equations. $W^{1,p}(\mathbb{R}^n)$ is the space of functions that are integrable to the $p$-th power and whose weak derivatives are also $L^p$-integrable.

The classical definition of Sobolev spaces is inherently local, relying on integrability and differentiability properties of functions. However, recent developments have introduced non-local characterizations of Sobolev spaces, which are

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based on function differences and integral expressions. These new approaches offer deeper insights and alternative perspectives on the structure of these spaces, which make them particularly useful in applications involving fractional Sobolev spaces, geometric measure theory, and the analysis of sets with a finite perimeter.

In [3], for $p \geq 1$ and $u \in L^p(\mathbb{R}^n)$, in order to provide a new characterization of the Sobolev space $W^{1,p}(\mathbb{R}^n)$, the authors introduced the functional

$$\Phi_\varepsilon(u,p) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^p}\rho_\varepsilon(|x-y|)\,dx\,dy, \tag{1.1}$$

where $\rho_\varepsilon$ is a sequence of radial mollifiers, that is, $\rho_\varepsilon \in L^1_{\rm loc}(0,\infty), \rho_\varepsilon \geq 0$ satisfying

$$\int_0^\infty \rho_\varepsilon(r) r^{n-1}\,dr = 1, \quad \forall \varepsilon > 0,$$

and for every $\delta > 0$,

$$\lim_{\varepsilon\to 0}\int_\delta^\infty \rho_\varepsilon(r) r^{n-1}\,dr = 0.$$

Furthermore, given $u \in L^p_{\rm loc}(\mathbb{R}^n)$, for any $\epsilon > 0$ we consider the following quantities:

$$k_\varepsilon(u,p) := \varepsilon^{n-p}\sup_{\mathcal{G}_\varepsilon}\sum_{Q'\in\mathcal{G}_\varepsilon} ⨍_{Q'}\left|u(x) - ⨍_{Q'} u\right|^p dx, \tag{1.2}$$

$$j_\varepsilon(u,p) = \varepsilon^{n-p}\sup_{\mathcal{G}_\varepsilon}\sum_{Q'\in\mathcal{G}_\varepsilon} ⨍_{Q'}⨍_{Q'} |u(x)-u(y)|^p\,dx\,dy, \tag{1.3}$$

where the supremum is taken over all families $\mathcal{G}_\varepsilon$ of disjoint $\varepsilon$-cubes $Q'$ of side length $\varepsilon$ and arbitrary orientation.

These functionals are closely related. Their relationships stem from their roles in measuring function differences and capturing essential features of Sobolev norms.

By setting $p=1$ and by taking $u=\chi_A$, where $\chi_A$ is the characteristic function of a measurable set $A$, the quantity $k_\varepsilon(u,1)$, introduced in [4], can be used to characterize sets of finite perimeter. The convergence of $k_\varepsilon$ as $\varepsilon \to 0$ was proved in [1], where the authors computed

$$\lim_{\varepsilon\to 0} k_\varepsilon(\chi_A,1) = \frac{1}{2}P(A).$$

Furthermore, setting $u=\chi_A$ in (1.1), a characterization of sets with finite perimeter (in the sense of De Giorgi) was obtained in [5] and

$$\lim_{\varepsilon\to 0}\int_A\int_{A^C} \frac{1}{|x-y|}\rho_\varepsilon(|x-y|)\,dx\,dy = K_n P(A),$$