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Smooth Densities of Stochastic Differential Equations Forced by Degenerate Stable Type Noises

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Abstract. Malliavin calculus for stochastic differential equations (SDEs) forced by degenerate stable like noises has been intensively studied recently, see e.g. [Hao, Peng and Zhang, J. Theoret. Probab. 34 (2021); Zhang, Ann. Probab. 42(5) (2012); Zhang, Ann. Probab. 45 (2017)]. In this paper, we derive a simple inequality as a replacement of Norris type lemma and use it to show that two families of degenerate SDEs with stable like noises admit smooth density functions.

AMS subject classifications: 6H07, 6H10

Key words: Malliavin calculus, smooth density, stochastic differential equation, degenerate stable type noise.

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1 Introduction

We are concerned with smooth densities for the degenerate stochastic differential equations forced by stable like noises as follows:

$$\begin{cases} \mathrm{d}X_t = a(X_t)\mathrm{d}t + B\mathrm{d}L_t, \\ X_0 = x, \end{cases} \tag{1.1}$$

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where $X_t \in \mathbb{R}^d$ for each $t \geq 0, x \in \mathbb{R}^d$ and the hypotheses of a, B, L_t will be stated below. Malliavin calculus for stochastic differential equations forced by degenerate stable like noises has been intensively studied recently, see e.g. [8, 20, 21], in which Norris type lemma plays a crucial role. In this paper, we derive a simple inequality as a replacement and use it to show that two families of degenerate SDEs with stable like noises admit smooth density functions. The crucial step is estimating the smallest eigenvalue of the simplified Malliavin matrix, which only uses some elementary facts of Poisson processes and undergraduate level ordinary differential equations. For more research in this direction, we refer to [1–7, 9–11, 13–15, 18, 19].

1.1 Some preliminary of Lévy processes

Denote $\mathbb{R}_0^d = \mathbb{R}^d \setminus \{0\}$. Let L_t be a pure jump process with càdlàg trajectories, it is well known that there exist a Poisson random measure N on $(\mathbb{R}_0^d \times \mathbb{R}^+, \mathcal{B}(\mathbb{R}_0^d \times \mathbb{R}^+))$ and a Lévy intensity measure ν on $(\mathbb{R}_0^d, \mathcal{B}(\mathbb{R}_0^d))$ associated to L_t such that

$$\begin{aligned} &\nu(\{0\}) = 0, \quad \int_{\mathbb{R}_0^d} (1 \wedge |z|^2) \nu(\mathrm{d}z) < \infty, \\ &L_t = \int_0^t \int_{|z| \leq 1} z \tilde{N}(\mathrm{d}z, \mathrm{d}s) + \int_0^t \int_{|z|>1} z N(\mathrm{d}z, \mathrm{d}s), \end{aligned} \tag{1.2}$$

where $\tilde{N}(\mathrm{d}z, \mathrm{d}s) = N(\mathrm{d}z, \mathrm{d}s) - \nu(\mathrm{d}z)\mathrm{d}s$. It is well known that the random measure N can be defined by: For all $A \in \mathcal{B}(\mathbb{R}_0^d)$,

$$N(A \times [0,t]) = \sum_{0 \leq s \leq t} \sharp\{L_s - L_{s-} : L_s - L_{s-} \in A\}.$$

Moreover, $N(A \times [0,t])$ satisfies a Poisson distribution with the intensity $\nu(A)t$, more precisely,

$$\mathbb{P}\big(N(A \times [0,t]) = k\big) = \frac{\big(\nu(A)t\big)^k}{k!} \mathrm{e}^{-\nu(A)t}, \quad k = 0, 1, 2, \ldots.$$

We shall use this easy relation frequently in the proof of our crucial Lemma 3.2 below.

Throughout this paper we assume that