Commun. Math. Anal. Appl.
doi: 10.4208/cmaa.2025-0016

Vol. **4**, No. 4, pp. 550-576
November 2025

Long Time Behavior of the Stochastic 2D Navier-Stokes Equations

Benedetta Ferrario[1,*] and Margherita Zanella[2]

[1] *Dipartimento di Scienze Economiche e Aziendali, Università di Pavia, 27100 Pavia, Italy.*
[2] *Department of Mathematics, Politecnico di Milano, 20133 Milano, Italy.*

Received 16 December 2024; Accepted 8 June 2025

Abstract. We review some basic results on existence and uniqueness of the invariant measure for the two-dimensional stochastic Navier-Stokes equations. A large part of the literature concerns the additive noise case; after revising these models, we consider our recent result [Ferrario and Zanella, Discrete Contin. Dyn. Syst. 44(1), 2024] with a multiplicative noise.

AMS subject classifications: 35Q30, 35R60, 60H30, 60H15

Key words: Two dimensional stochastic Navier-Stokes equations, invariant measure, multiplicative noise, degenerate noise.

1 Introduction

The Navier-Stokes equations describe the motion of homogeneous incompressible viscous fluids, they are

$$\partial_t u - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f, \quad \mathrm{div} u = 0, \tag{1.1}$$

where $u=u(t,\xi)$ and $p=p(t,\xi)$ are the velocity vector and the (scalar) pressure, respectively, defined for $t\geq 0$ and $\xi\in\mathcal{O}\subseteq\mathbb{R}^d$ ($d=2$ or $d=3$), $\nu>0$ is the kinematic

*Corresponding author. *Email addresses:* benedetta.ferrario@unipv.it (B. Ferrario), margherita.zanella@polimi.it (M. Zanella)

viscosity parameter. Suitable initial and boundary conditions are given. In the right-hand side, the forcing term can be deterministic and/or stochastic.

In this paper we review some results for the bidimensional Navier-Stokes equations with both forcing terms, deterministic and stochastic. The literature is quite huge and we cannot quote all the contributions. Our purpose is to present and compare some results obtained in the last thirty years on the stationary solutions, called invariant measures in this setting. In particular, the results for the uniqueness of the invariant measure. Under some assumptions on the forcing terms, we will show that there exists a unique invariant measure even when the deterministic Navier-Stokes equations (1.1) have more than one stationary solution.

In the deterministic setting, uniqueness of the stationary solutions occurs when the data are small enough, or the viscosity is large enough (see [38, Chapter 2]). Roughly speaking, a strong enough dissipation prevents the existence of different stationary solutions. Moreover, when the stationary solution is unique then the system converges to it as time diverges to $+\infty$.

We point out that the behavior of the motion of fluids with small viscosity is a interesting and challenging problem, also in connection with the vanishing viscosity limit leading to the equation of motion of inviscid fluids (i.e. the Euler equations).

The idea proposed by Kolmogorov (see [19,20,40] and reference therein) in his K41 theory of turbulence is to introduce a noisy forcing term in the equation of motion so to mix up the dynamics and break the possibility of different asymptotic behaviors. In this setting a statistical description is given, as usually done when describing the chaotic behaviour of the so called turbulent flows. By means of an invariant (stationary) probability measure, when it exists and is unique, one can describe the asymptotic behavior of a turbulent fluid from a statistical point of view. Therefore, the stochastic Navier-Stokes equations are an important model in the turbulence theory of fluids.

Starting from this idea, many researchers have studied the Navier-Stokes equations with a stochastic forcing term. In this paper we revise some contributions for the Navier-Stokes equations in a smooth bounded domain of $\mathbb{R}^2$, where in the right-hand side of the equation of motion (1.1) we introduce a noisy forcing term $G(u)\mathrm{d}W$. It represents a noise which is white in time and colored in space; moreover it might depend on the velocity (the so called multiplicative noise) or not (the so called additive noise). We will focus mainly on our recent result, when the noise is multiplicative and of at most linear growth. But we will revise also the main results for the case of additive noise, as well. In the literature there are also different models with a random force acting at discrete times (see, e.g. [1,32]).