Commun. Math. Anal. Appl.
doi: 10.4208/cmaa.2025-0017

Vol. **4**, No. 4, pp. 577-593
November 2025

A Stochastic Maximum Principle for Relaxed Control with General Risk Measure and Its Application in Finance

Shuaiqi Zhang[1,*], Jie Xiong[2] and Xin Zhang[3]

[1] *School of Mathematics, China University of Mining and Technology, Xuzhou 221116, China.*

[2] *Department of Mathematics, and SUSTech International Center for Mathematics, Southern University of Science and Technology, Shenzhen 518055, China.*

[3] *School of Mathematics, Southeast University Nanjing, Nanjing 211189, China.*

Received 22 December 2024; Accepted 8 September 2025

Abstract. This paper deals with optimal relaxed control problem where the cost functional is a general risk measure instead of an expectation. We develop a stochastic maximum principle for this kind of optimal control problems using a variational method. Then, under the expectation optimization objective, a dynamic programming principle is studied and its connections to the adjoint process is shown. At last, the result is applied to two examples. One is a linear quadratic problem and the other is an optimal investment problem.

AMS subject classifications: 93E03, 93E20

Key words: Measure valued control process, stochastic maximum principle.

---

1 Introduction

Since the pioneering works of Pontryagin and his colleagues in the 1950s, the Pontryagin maximum principle was extensively applied to study both the deterministic and stochastic optimal control problems. The basic idea of Pontryagin maximum principle is to derive the necessary or sufficient conditions which

*Corresponding author. *Email address:* `shuaiqiz@hotmail.com` (S. Zhang)

must be satisfied by the optimal control. The stochastic maximum principle is a major extension of Pontryagin maximum principle for solving stochastic control problems. [24–26] were the first to derive the stochastic maximum principle for stochastic control problems with only terminal fixed time cost. Specially, the methods of spike variation and Neustadt's variational principle (see [30]) are used in [26] for deriving the stochastic maximum principle. The stochastic maximum principle for the cases that the control enter the diffusion term or not was derived by [8] for stochastic control problems with both running and terminal cost. However, the convexity of the control domain is needed for obtaining the stochastic maximum principle when the control enter the diffusion term. By introducing the first and second order variational inequality, [34] finally obtain the stochastic maximum principle when the control domain need not to be convex and the diffusion term contains the control variable. Since the work of Peng, versions of stochastic maximum principle were extensively investigated in more general stochastic systems, see for example, [10, 23, 31, 33, 36, 37, 40], and so on.

Besides the stochastic maximum principle, dynamic programming approach is another important method for dealing with stochastic control problems. The dynamic programming technique was first introduced by [6, 7] in 1950s. Since then, a number of scholars entered this field and the dynamic programming approach was widely used in solving stochastic control problems under different stochastic systems, see [19, 22, 28, 35, 39, 41] and so on. Interested readers may also refer to the monographs of [16, 32, 42] for a complete treatment of the dynamic programming approach. In addition, the relationship between dynamic programming and stochastic maximum principle under different stochastic models was studied by [2, 12, 13, 17, 38, 43].

Most of the above mentioned works focused on the strict (or classical) stochastic control problems, i.e. the control u_t at time t is merely an element of the control action space $\mathbb{U}$. Under some convexity conditions, the existence of the strict optimal control was investigated by [20, 21, 27]. However, the strict control problems may fail to exist an optimal solution without convexity conditions. To overcome this problem, the relaxed control was introduced by embedding the set of strict controls into a larger set of probability measures on control action space $\mathbb{A}$. For the relaxed control, the controller at time t chooses a probability measure $\pi_t(da)$ on the control action space $\mathbb{A}$, instead of an element u_t of $\mathbb{A}$. Relaxed control, in other words, is a measure-valued control process, which can be seen as generalization version of the strict control. In fact, if $\pi_t(da)$ is a Dirac measure concentrated at a single point u_t, then the relaxed control degenerates to a strict control.

There are quite many interest in studying relaxed control problem, see [1, 3–5, 14, 15, 29]. In detail, [15] first proved the existence of optimal relaxed control