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Strong Convergence of Functional Stochastic Differential Equations via the Functional Itô Calculus

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Abstract. In general, Malliavin calculus is exploited to construct the Milstein scheme for functional stochastic differential equations due to the appearance of anticipative stochastic integrals. In this paper, by passing the Malliavin calculus, Milstein-type schemes are constructed via the functional Itô calculus for a range of functional stochastic differential equations. As regards the Milstein schemes designed, the strong convergence with the rate 1 is established for functional stochastic differential equations, where the delay measure need not be absolutely continuous with respect to the Lebesgue measure.

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Key words: Functional Itô calculus, Milstein scheme, functional stochastic differential equation, convergence rate, strong convergence.

1 Introduction

Many physical phenomena can be modelled by stochastic dynamical systems whose evolution in time-dependent states on a finite part of its past history. Such models may be identified as stochastic functional differential equations (SFDEs) or stochastic delay differential equations (SDDEs). There are many forms of

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SDDEs, for example, the Itô delay stochastic differential equation (SDE)

$$\mathrm{d}X(t) = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W(t),$$

where $(W(t))_{t\geq 0}$ is a d-dimensional Brownian motion. If b and σ are taken as projection functions, we obtain a point-delay SDE immediately. Another example is distribution delay SDEs, i.e. stochastic differential equation with distributed memory term like $\int_{-\tau}^{0} X(t+s)^{-}(ds)$, where μ is a probability measure on $[-\tau,0]$. It is important to note that the segment process is defined in an infinite-dimensional space, making the study of its derivative estimates quite challenging. Whereas, for equations of the distributed delay type, since their Dupire's horizontal and vertical derivatives can be computed, the functional Itô formula provides a powerful tool for addressing many problems.

In recent years, SFDEs have been used in a wide spectrum of applications [7–9]. The simulation of stochastic delay differential equation mainly contains the following two common methods, one of which is the Euler scheme introduced in [6] and the strong convergence to the solution of the SDDEs, of order $1/2$ in the time-step. The result in [6] is improved to order 1 in [4]. In this paper, we are concerned with the higher-order numerical schemes of order 1 named Milstein scheme that is another numerical simulation methods. Recently, there has been a lot of works on the development of the Milstein schemes for SDDEs which include point-delay and distribution delay, etc. Hu *et al.* [4] studied the Milstein schemes for stochastic point-delay differential equation using Malliavin calculus and the anticipated stochastic analysis. This work is more complicated and the equation is just point-delay which is not common enough. Further, Kloeden and Shardlow [5] considered the stochastic segment process-delay differential equation which is not involve anticipative integrals and anticipative stochastic calculus. He just used Taylor expansions and calculates the local truncation error by collecting the Taylor remainder terms. Although it has less calculations compared to [4], it required high regularity of the equation (at least C^3) and was not good for computer simulations. In addition, Buckwar [1] studied the Milstein approximations for SFDE with distributed memory term i.e. the drift and diffusion includes the distribution-delay term. The result in [1] is not perfect in the order of the Milstein scheme is close to 1 but not 1. In the second work of our paper, we use the better tool (functional Itô formula) to reach the order 1.

For numerical simulation of SFDEs, as we all aware that anticipation analysis plays an indispensable role but it makes the calculation more difficult. Dupire [3] generalized the Itô formula to a functional setting by using pathwise functional derivatives. The functional Itô formula enables us to consider a larger class of stochastic systems with any form delays and calculate it directly without Malli-