Commun. Math. Anal. Appl.
doi: 10.4208/cmaa.2025-0019

Vol. **4**, No. 4, pp. 626-646
November 2025

Recent Progress on Stochastic Fractional Diffusion Equations with Space-Time White Noise

Yuhui Guo and Jiang-Lun Wu*

Guangdong Provincial Key Laboratory of IRADS, Department of Mathematical Sciences, Beijing Normal – Hong Kong Baptist University, Zhuhai 519087, China.

Received 25 February 2025; Accepted 24 May 2025

Abstract. We review the recent development on stochastic fractional diffusion equations with space-time white noise. We explicate the Fox H-function, fundamental solutions, mild Skorohod solution and its properties including existence and uniqueness, intermittent property and Hölder continuity.

AMS subject classifications: Primary: 60H15; Secondary: 60H40, 33E20, 37H15, 26B35

Key words: Stochastic partial differential equation, space-time white noise, Fox H-function, intermittency, Hölder continuity.

1 Introduction

Stochastic fractional diffusion equations (stochastic FDEs) with Gaussian noise have been extensively studied over the last decade, see e.g. [6, 11–15, 17, 26, 34, 37] and references therein. Due to the nonlocal nature of fractional differentiation respectively with time and with space variables, stochastic FDEs have received a lot of attention and been applied to describing more complicated physical phenomena, especially viscoelasticity, see for instance [18, 39]. A material is said to be viscoelastic if the material has an elastic (recoverable) part as well as a vis-

*Corresponding author. *Email addresses:* `yuhuiguo@uic.edu.cn` (Y. Guo), `jianglunwu@uic.edu.cn` (J.-L. Wu)

cous (nonrecoverable) part. In 2010, Mainardi put these two properties together and turned them into a linear equation by using fractional calculus, see [36] for more detailed discussions. Since then, such kind of equations has been widely applied to many diverse research fields in physics [32], chemistry [24, 25], and biology [35], just mention a few.

In this survey article, we will present two important properties of the solutions to stochastic FDEs with space-time white noise, namely, intermittent property and Hölder continuity.

The concept of intermittency property arises from physics in which physicists heuristically describe that a system is called intermittent if it grows a few large and steep peaks concentrated on small islands when time is sufficiently long. Bakry *et al.* [1] and Zel'dovich *et al.* [47] established a mathematically rigorours definition of intermittency in the context of stochastic partial differential equations, which is explicated as follows. For a random field

$$u=\{u(t,x): t\geq 0, x\in\mathbb{R}^d\},$$

one can define the upper and lower (moment) Lyapunov exponents, respectively, as

$$\begin{aligned} \overline{L}_p &:= \limsup_{t\to\infty}\frac{\log\mathbb{E}[|u(t,x)|^p]}{t}, \\ \underline{L}_p &:= \liminf_{t\to\infty}\frac{\log\mathbb{E}[|u(t,x)|^p]}{t} \end{aligned} \tag{1.1}$$

for $p\in\mathbb{N}$. If the above two limits coincide, we then denote $L_p := \overline{L}_p = \underline{L}_p$ and call it the p-th (moment) Lyapunov exponent. Moreover, a random field $u(t,x)$ is said to be intermittent if $p\mapsto L_p/p$ is strictly increasing, see [5]. For a detailed explanation of the fact, in the case of the stochastic heat equations, that strictly increasing property of this function do imply the existence of high peaks when t is sufficiently large, the interested reader is referred to [4, 20].

The property of Hölder continuity for stochastic FDEs was first investigated by [15], in which the following completely monotonic property of the two-parameter Mittag-Leffler function:

$$x\in[0,\infty) \;\mapsto\; E_{a,b}(-x) \quad \text{is complete monotone} \;\Longleftrightarrow\; 0<a\leq 1\wedge b$$

has been applied in a crucial way. This property restricts the results therein only to the case when the time fractional order index $\beta\in(0,1]$. The corresponding result has been extended to the case of $\beta\in(0,2)$ via the techniques of local fractional derivative and fractional Taylor expansion in [13]. Then, Hölder continuity for all