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Exponential Ergodicity for Time-Periodic McKean-Vlasov SDEs

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Abstract. As extensions to the corresponding results derived for time homogeneous McKean-Vlasov SDEs, the exponential ergodicity is proved for time-periodic distribution dependent SDEs in three different situations:

1) in the quadratic Wasserstein distance and relative entropy for the dissipative case;
2) in the Wasserstein distance induced by a cost function for the partially dissipative case; and
3) in the weighted Wasserstein distance induced by a cost function and a Lyapunov function for the fully non-dissipative case.

The main results are illustrated by time inhomogeneous granular media equations, and are extended to reflecting McKean-Vlasov SDEs in a convex domain.

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Key words: Time-periodic McKean-Vlasov SDE, exponential erodicity, relative entropy, Wasserstein distance.

1 Introduction

By using the log-Harnack and Talagrand inequalities, the exponential ergodicity in relative entropy was proved in [8] for a class of McKean-Vlasov SDEs, which

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include as typical examples the granular porous media equations investigated in [4, 6]. Next, by using coupling methods, the exponential ergodicity in different probability metrics has been derived in [12] for partially dissipative and non-dissipative models. The techniques involving Lyapunov-type functions V and associated weighted Wasserstein distances have been employed in infinite-dimensional distribution-dependent models to characterize the "largeness" of the noise, as illustrated in [9]. Moreover, these types of exponential ergodicity have been investigated in [13] for reflecting McKean-Vlasov SDEs. In this paper, we extend these results to time-periodic (reflecting) McKean-Vlasov SDEs.

Let $D\subset\mathbb{R}^d$ be a convex domain. When $D\neq\mathbb{R}^d$, it has a non-empty boundary ∂D. In this case, for any $x\in\partial D$ and $r>0$, let

$$\mathscr{N}_{x,r}:=\{\mathbf{n}\in\mathbb{R}^d:|\mathbf{n}|=1, B(x-r\mathbf{n},r)\cap D=\emptyset\},$$

where $B(x,r):=\{y\in\mathbb{R}^d:|x-y|<r\}$. We have

$$\mathscr{N}_x:=\cup_{r>0}\mathscr{N}_{x,r}\neq\emptyset,\quad x\in\partial D.$$

We call $\mathscr{N}_x$ the set of inward unit normal vectors of ∂D at point x. Since D is convex, $\mathscr{N}_x\neq\emptyset$ for $x\in\partial D$ and

$$\langle x-y,\mathbf{n}(x)\rangle\leq 0,\quad y\in\bar{D},\quad x\in\partial D,\quad \mathbf{n}(x)\in\mathscr{N}_x. \tag{1.1}$$

Let $\mathscr{P}(\bar{D})$ be the space of all probability measures on the closure $\bar{D}$ of D, equipped with the weak topology. Consider the following reflecting McKean-Vlasov SDE on $\bar{D}\subset\mathbb{R}^d$:

$$\mathrm{d}X_t=b_t(X_t,\mathscr{L}_{X_t})\mathrm{d}t+\sigma_t(X_t,\mathscr{L}_{X_t})\mathrm{d}W_t+\mathbf{n}(X_t)\mathrm{d}l_t,\quad t\geq 0, \tag{1.2}$$

where W_t is an m-dimensional Brownian motion on a complete filtration probability space $(\Omega,\{\mathscr{F}_t\}_{t\geq 0},\mathbb{P})$, $\mathscr{L}_{X_t}$ is the distribution of X_t, $\mathbf{n}(x)\in\mathscr{N}_x$ for $x\in\partial D$, l_t is an adapted increasing process which increases only when $X_t\in\partial D$, and

$$\begin{aligned} b:[0,\infty)\times\mathbb{R}^d\times\mathscr{P}(\bar{D}) &\to \mathbb{R}^d,\\ \sigma:[0,\infty)\times\mathbb{R}^d\times\mathscr{P}(\bar{D}) &\to \mathbb{R}^d\otimes\mathbb{R}^m \end{aligned}$$

are measurable. When $D=\mathbb{R}^d$, we simply denote $\mathscr{P}=\mathscr{P}(\bar{D})$. In this case, we have $\partial D=\emptyset$, so that $l_t=0$ and (1.2) reduces to

$$\mathrm{d}X_t=b_t(X_t,\mathscr{L}_{X_t})\mathrm{d}t+\sigma_t(X_t,\mathscr{L}_{X_t})\mathrm{d}W_t,\quad t\geq 0. \tag{1.3}$$

The SDE (1.2) or (1.3) is called strong (weak) well-posed for distributions in a subspace $\hat{\mathscr{P}}\subset\mathscr{P}(\bar{D})$, if for any $s\geq 0$ and any $\mathscr{F}_s$-measurable variable X_s with