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Exponential Ergodicity for Time-Periodic McKean-Vlasov SDEs

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Abstract. As extensions to the corresponding results derived for time homogeneous McKean-Vlasov SDEs, the exponential ergodicity is proved for time-periodic distribution dependent SDEs in three different situations:

1) in the quadratic Wasserstein distance and relative entropy for the dissipative case;
2) in the Wasserstein distance induced by a cost function for the partially dissipative case; and
3) in the weighted Wasserstein distance induced by a cost function and a Lyapunov function for the fully non-dissipative case.

The main results are illustrated by time inhomogeneous granular media equations, and are extended to reflecting McKean-Vlasov SDEs in a convex domain.

AMS subject classifications: 60B05, 60B10
Key words: Time-periodic McKean-Vlasov SDE, exponential erodicity, relative entropy, Wasserstein distance.

1 Introduction

By using the log-Harnack and Talagrand inequalities, the exponential ergodicity in relative entropy was proved in [8] for a class of McKean-Vlasov SDEs, which

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include as typical examples the granular porous media equations investigated in [4, 6]. Next, by using coupling methods, the exponential ergodicity in different probability metrics has been derived in [12] for partially dissipative and non-dissipative models. The techniques involving Lyapunov-type functions V and associated weighted Wasserstein distances have been employed in infinite-dimensional distribution-dependent models to characterize the "largeness" of the noise, as illustrated in [9]. Moreover, these types of exponential ergodicity have been investigated in [13] for reflecting McKean-Vlasov SDEs. In this paper, we extend these results to time-periodic (reflecting) McKean-Vlasov SDEs.

Let $D\subset\mathbb{R}^d$ be a convex domain. When $D\neq\mathbb{R}^d$, it has a non-empty boundary ∂D. In this case, for any $x\in\partial D$ and $r>0$, let

$$\mathscr{N}_{x,r}:=\{\mathbf{n}\in\mathbb{R}^d:|\mathbf{n}|=1,B(x-r\mathbf{n},r)\cap D=\emptyset\},$$

where $B(x,r):=\{y\in\mathbb{R}^d:|x-y|<r\}$. We have

$$\mathscr{N}_x:=\cup_{r>0}\mathscr{N}_{x,r}\neq\emptyset,\quad x\in\partial D.$$

We call $\mathscr{N}_x$ the set of inward unit normal vectors of ∂D at point x. Since D is convex, $\mathscr{N}_x\neq\emptyset$ for $x\in\partial D$ and

$$\langle x-y,\mathbf{n}(x)\rangle\leq 0,\quad y\in\bar{D},\quad x\in\partial D,\quad \mathbf{n}(x)\in\mathscr{N}_x. \tag{1.1}$$

Let $\mathscr{P}(\bar{D})$ be the space of all probability measures on the closure $\bar{D}$ of D, equipped with the weak topology. Consider the following reflecting McKean-Vlasov SDE on $\bar{D}\subset\mathbb{R}^d$:

$$\mathrm{d}X_t=b_t(X_t,\mathscr{L}_{X_t})\mathrm{d}t+\sigma_t(X_t,\mathscr{L}_{X_t})\mathrm{d}W_t+\mathbf{n}(X_t)\mathrm{d}l_t,\quad t\geq 0, \tag{1.2}$$

where W_t is an m-dimensional Brownian motion on a complete filtration probability space $(\Omega,\{\mathscr{F}_t\}_{t\geq 0},\mathbb{P})$, $\mathscr{L}_{X_t}$ is the distribution of X_t, $\mathbf{n}(x)\in\mathscr{N}_x$ for $x\in\partial D$, l_t is an adapted increasing process which increases only when $X_t\in\partial D$, and

$$\begin{aligned} b:[0,\infty)\times\mathbb{R}^d\times\mathscr{P}(\bar{D}) &\to \mathbb{R}^d,\\ \sigma:[0,\infty)\times\mathbb{R}^d\times\mathscr{P}(\bar{D}) &\to \mathbb{R}^d\otimes\mathbb{R}^m \end{aligned}$$

are measurable. When $D=\mathbb{R}^d$, we simply denote $\mathscr{P}=\mathscr{P}(\bar{D})$. In this case, we have $\partial D=\emptyset$, so that $l_t=0$ and (1.2) reduces to

$$\mathrm{d}X_t=b_t(X_t,\mathscr{L}_{X_t})\mathrm{d}t+\sigma_t(X_t,\mathscr{L}_{X_t})\mathrm{d}W_t,\quad t\geq 0. \tag{1.3}$$

The SDE (1.2) or (1.3) is called strong (weak) well-posed for distributions in a subspace $\hat{\mathscr{P}}\subset\mathscr{P}(\bar{D})$, if for any $s\geq 0$ and any $\mathscr{F}_s$-measurable variable X_s with

$\mathscr{L}_{X_s}\in\hat{\mathscr{P}}$, (1.2) has a unique strong (weak) solution $(X_t)_{t\geq s}$ with $\mathscr{L}_{X_\cdot}\in C([s,\infty);\hat{\mathscr{P}})$, the space of continuous maps from $[s,\infty)$ to $\hat{\mathscr{P}}$ under the weak topology. In this case, we denote $P^*_{s,t}\mu=\mathscr{L}_{X_t}$ for the solution with $\mathscr{L}_{X_s}=\mu\in\hat{\mathscr{P}}$. We call the SDE well-posed, if it is both strong and weak well-posed.

In this paper, we investigate the exponential ergodicity of (1.2) and (1.3) with t_0-periodic coefficients for some $t_0>0$,

$$(b_{t+t_0},\sigma_{t+t_0})=(b_t,\sigma_t),\quad t\geq 0$$

such that the corresponding results derived in [8, 12, 13] are extended to time inhomogeneous models. By the t_0-periodicity and the well-posedness for distributions in $\hat{\mathscr{P}}$, we have

$$P^*_{s,t}\mu=P^*_{s+nt_0,t+nt_0}\mu,\quad t\geq s\geq 0,\quad n\in\mathbb{N},\quad \mu\in\hat{\mathscr{P}}. \tag{1.4}$$

In this case, a probability measure $\bar\mu_0\in\hat{\mathscr{P}}$ is called an invariant probability measure, if $P^*_{0,t_0}\bar\mu_0=\bar\mu_0$. Combining this with (1.4), we see that the measures

$$\bar\mu_s:=P^*_{0,s}\bar\mu_0,\quad s\in[0,t_0]$$

satisfy

$$P^*_{s+mt_0,s+(m+n)t_0}\bar\mu_s=\bar\mu_s,\quad n,m\in\mathbb{Z}_+,\quad s\in[0,t_0]. \tag{1.5}$$

Let $\mathbb{W}:\hat{\mathscr{P}}\times\hat{\mathscr{P}}\to[0,\infty)$ with $\mathbb{W}(\mu,\nu)=0$ if and only if $\mu=\nu$. We call (1.2) exponential ergodic in $\mathbb{W}$, if there exist constants $c,\lambda>0$ such that

$$\mathbb{W}\big(P^*_{s,s+nt_0}\mu,\bar\mu_s\big)\leq c\mathrm{e}^{-\lambda n}\mathbb{W}(\mu,\bar\mu_s),\quad n\in\mathbb{N},\quad \mu\in\hat{\mathscr{P}},\quad s\in[0,t_0]. \tag{1.6}$$

By (1.4), this is equivalent to

$$\mathbb{W}\big(P^*_{s+mt_0,s+(m+n)t_0}\mu,\bar\mu_s\big)\leq c\mathrm{e}^{-\lambda n}\mathbb{W}(\mu,\bar\mu_s),\quad n,m\in\mathbb{Z}_+,\quad \mu\in\hat{\mathscr{P}},\quad s\in[0,t_0].$$

So, we will only consider (1.6).

The remainder of the paper is organized as follows. In Sections 2-4, we study the exponential ergodicity for (1.3) without reflection, where Section 2 considers dissipative models for $\mathbb{W}$ being the quadratic Wasserstein distance $\mathbb{W}_2$ or the relative entropy $\mathbb{H}$, Section 3 concerns with partially dissipative models with $\mathbb{W}=\mathbb{W}_\psi$ induced by a cost function ψ, and Section 4 deals with fully non-dissipative models for $\mathbb{W}=\mathbb{W}_{\psi,V}$ induced by a cost function ψ and a Lyapunov function V. Finally, these results are extended in Section 5 to the reflecting SDE (1.2) on a convex domain D.

2 Exponential ergodicity in relative entropy and $\mathbb{W}_2$

Corresponding to [4,6,8] where the exponential ergodicity in entropy is investigated in the time homogeneous case, we consider the exponential ergodicity in relative entropy for (1.3). Recall that the relative entropy for probability measures $\mu_1,\mu_2\in\mathscr{P}$ is given by

$$\mathbb{H}(\mu_1|\mu_2):=\begin{cases}\mu_2(\rho\log\rho), & \text{if} \quad \rho:=\dfrac{\mathrm{d}\mu_1}{\mathrm{d}\mu_2} \quad \text{exists},\\ \infty, & \text{otherwise}.\end{cases}$$

Let $\mu(f):=\int f\mathrm{d}\mu$ for a measure μ and $f\in L^1(\mu)$. For the symmetric diffusion process generated by $L:=\Delta+\nabla V$ on $\mathbb{R}^d$ with $\bar\mu(\mathrm{d}x):=\mathrm{e}^{V(x)}\mathrm{d}x\in\mathscr{P}$, the exponential ergodicity in $\mathbb{H}$ with rate $\lambda>0$, i.e. the associated diffusion semigroup P_t satisfies

$$\bar\mu\big((P_tf)\log(P_tf)\big)\le\mathrm{e}^{-\lambda t}\bar\mu(f\log f),\quad t\ge0,\quad f\ge0,\quad \bar\mu(f)=1,$$

if and only if the following log-Sobolev inequality holds:

$$\bar\mu(f^2\log f^2)\le\frac{4}{\lambda}\bar\mu(|\nabla f|^2),\quad f\in C_b^1(\mathbb{R}^d),\quad \bar\mu(f^2)=1,$$

see for instance [2].

According to the concentration property of the log-Sobolev inequality (see [1]), there exists $\varepsilon>0$ such that $\bar\mu(\mathrm{e}^{\varepsilon|\cdot|^2})<\infty$, so that by Young's inequality, $\mathbb{H}(\mu|\bar\mu)<\infty$ implies

$$\mu(|\cdot|^2)\le\varepsilon^{-1}\big\{\mathbb{H}(\mu|\bar\mu)+\log\bar\mu\big(\mathrm{e}^{\varepsilon|\cdot|^2}\big)\big\}<\infty.$$

Therefore, to investigate the exponential convergence in entropy, it is natural to consider distributions in the Wasserstein space

$$\mathscr{P}_2:=\{\mu\in\mathscr{P}:\mu(|\cdot|^2)<\infty\},$$

which is a Polish space under the quadratic Wasserstein distance

$$\mathbb{W}_2(\mu_1,\mu_2):=\inf_{\pi\in\mathscr{C}(\mu_1,\mu_2)}\left(\int_{\mathbb{R}^d\times\mathbb{R}^d}|x-y|^2\pi(\mathrm{d}x,\mathrm{d}y)\right)^{\frac{1}{2}},\quad \mu_1,\mu_2\in\mathscr{P}_2,$$

where $\mathscr{C}(\mu_1,\mu_2)$ is the set of all couplings of μ_1 and μ_2.

2.1 Assumptions

Let δ_x be the Dirac measure at $x\in\mathbb{R}^d$. We assume

(H_1) $|b_t(0,\delta_0)|+\|\sigma_t(0,\delta_0)\|$ is locally integrable in $t\ge 0$, and there exist $K_1,K_2,K_3\in L^1_{loc}([0,\infty);\mathbb{R})$ such that

$$\begin{aligned}&\|\sigma_t(x,\mu)-\sigma_t(y,\nu)\|^2\le K_3(t)\big(|x-y|^2+\mathbb{W}_2(\mu,\nu)^2\big),\\&2\langle b_t(x,\mu)-b_t(y,\nu),x-y\rangle+\|\sigma_t(x,\mu)-\sigma_t(y,\nu)\|_{HS}^2\\&\le K_1(t)|x-y|^2+K_2(t)\mathbb{W}_2(\mu,\nu)^2,\quad t\ge 0,\quad x,y\in\mathbb{R}^d,\quad \mu,\nu\in\mathscr{P}_2.\end{aligned}$$

According to [7, Theorem 3.3] (see also [11]), under this condition the SDE (1.3) is well-posed for distributions in $\mathscr{P}_2$, and

$$\mathbb{W}_2(P_{s,t}^*\mu,P_{s,t}^*\nu)^2\le \mathrm{e}^{\int_s^t(K_1(r)+K_2(r))\mathrm{d}r}\mathbb{W}_2(\mu,\nu)^2,\quad t\ge s\ge 0,\quad \mu,\nu\in\mathscr{P}_2. \tag{2.1}$$

To deduce from (2.1) the exponential ergodicity in entropy, we need the following condition.

(H_2) $\sigma_t(x,\mu)=\sigma_t(x)$ does not depend on μ and is invertible, and there exist increasing positive measurable functions $\lambda,\kappa_1,\kappa_2$ such that

$$\begin{aligned}&2\langle b_t(x,\mu)-b_t(y,\nu),x-y\rangle^+ +\|\sigma_t(x)-\sigma_t(y)\|_{HS}^2\\&\le\kappa_1(t)|x-y|^2+\kappa_2(t)|x-y|\mathbb{W}_2(\mu,\nu),\\&\big\|\sigma_t(x)^{-1}\big\|\le\lambda(t),\quad t\ge 0,\quad x,y\in\mathbb{R}^d,\quad \mu,\nu\in\mathscr{P}_2.\end{aligned}$$

Obviously, (H_2) implies (H_1) for $K_1(t)=\kappa_1(t)+\beta_t$ and $K_2(t)=\kappa_2(t)^2/(4\beta_t)$ for $\beta_t>0$, but in applications we may take better choices of (K_1,K_2) than that implied by (H_2). For any $t\ge s\ge 0$, let

$$\lambda(s,t):=\sup_{r\in[s,t]}\lambda(r),\quad \kappa_i(s,t):=\sup_{r\in[s,t]}\kappa_i(r),\quad i=1,2.$$

We intend to establish the following type of estimate:

$$\mathbb{H}(P_{s,t}^*\mu|P_{s,t}^*\nu)\le\phi(s,t)\mathbb{H}(\mu|\nu),\quad t>s,\quad \mu\in\mathscr{P}_2 \tag{2.2}$$

for a reasonable class of measures $\nu\in\mathscr{P}_2$. In the time homogeneous situation, one takes ν as the invariant probability measures so that $\mathbb{P}_{s,t}^*\nu=\nu$ for all $t\ge s$.

As explained above, to derive (2.2), we need to establish the log-Sobolev inequality for $P_{s,t}^*\nu$. To this end, we apply the Bakry-Emery curvature for the associated time-distribution dependent generator of (1.3)

$$L_{t,\mu}:=\frac{1}{2}\mathrm{tr}\{\sigma_t\sigma_t^*\nabla^2\}+b_t(\cdot,\mu)\cdot\nabla,\quad t\geq 0,\quad \mu\in\mathscr{P}_2.$$

According to [5], we introduce

$$\begin{aligned}&\Gamma_t^1(f,g):=\frac{1}{2}\langle\sigma_t\sigma_t^*\nabla f,\nabla g\rangle,\quad f,g\in C^1(\mathbb{R}^d),\\&\Gamma_{t,\mu}^2(f,f):=\frac{1}{2}L_{t,\mu}\Gamma_t^1(f,f)-\Gamma_t^1(f,L_{t,\mu}f)+\frac{1}{2}\partial_t\Gamma_t^1(f,f),\quad f\in C^3(\mathbb{R}^d).\end{aligned}\tag{2.3}$$

Note that the above carré du champ is different from that in [4] where derivatives in t is also involved. To make $\Gamma_{t,\mu}^2$ meaningful and also for late use, we assume

(H_3) $A_t:=\|\sigma_t\|_\infty\in L_{loc}^2([0,\infty))$, at least one of the following two conditions holds:

(1) σ_t is constant for each $t\geq 0$;

(2) $\sigma_t(x)$ is C^1 in t and C^2 in x, $b_t(x)$ is C^1 in x, and there exists a function $\gamma\in L_{loc}^1([0,\infty);\mathbb{R})$ such that

$$\Gamma_{t,\mu}^2(f,f)\geq\gamma_t\Gamma_t^1(f,f),\quad t\geq 0,\quad f\in C^3(\mathbb{R}^d),\quad \mu\in\mathscr{P}_2.$$

Finally, for any constant $c>0$, we write $\nu\in T_c$ if $\nu\in\mathscr{P}$ satisfying the Talagrand inequality

$$\mathbb{W}_2(\mu,\nu)^2\leq c\mathbb{H}(\mu|\nu).\tag{2.4}$$

According to [3], this inequality is implied by the log-Sobolev inequality

$$\nu(f^2\log f^2)\leq c\nu(|\nabla f|^2),\quad f\in C_b^1(\mathbb{R}^d),\quad \nu(f^2)=1,\tag{2.5}$$

for which we denote $\nu\in Log_c$.

2.2 Main results

Theorem 2.1. *Assume (H_1) and that (1.3) is t_0-periodic for some $t_0>0$ with*

$$\lambda:=-\int_0^{t_0}\{K_1(r)+K_2(r)\}\mathrm{d}r>0.\tag{2.6}$$

(1) (1.3) *has a unique invariant probability measure* $\bar{\mu}_0$ *such that*

$$\mathbb{W}_2(P^*_{nt_0}\mu,\bar{\mu}_0)^2 \le \mathrm{e}^{-n\lambda}\mathbb{W}_2(\mu,\bar{\mu}_0)^2, \quad \mu\in\mathscr{P}_2, \quad n\in\mathbb{N}. \tag{2.7}$$

(2) *If* (H_2), *and one of* $(H_3)(1)$ *or* $(H_3)(2)$ *with* $\int_0^{t_0}\gamma_s\mathrm{d}s>0$ *hold, then there exists a constant* $c>0$ *such that for any* $n\in\mathbb{N}$ *and* $\mu\in\mathscr{P}_2$,

$$\max\big\{\mathbb{H}\big(P^*_{nt_0}\mu|\bar{\mu}_0\big),\mathbb{W}_2\big(P^*_{nt_0}\mu,\bar{\mu}_0\big)^2\big\}\le c\mathrm{e}^{-\lambda n}\min\big\{\mathbb{H}(\mu|\bar{\mu}_0),\mathbb{W}_2(\mu,\bar{\mu}_0)^2\big\}. \tag{2.8}$$

To illustrate this result, we consider the time-dependent version of granular media equations studied in [4,6,8]. Let $V\in C^{0,2}([0,\infty)\times\mathbb{R}^d)$ and $W\in C^{0,2}([0,\infty)\times\mathbb{R}^{2d})$ such that

$$\int_{\mathbb{R}^d}\mathrm{e}^{-V_t(x)}\mathrm{d}x+\int_{\mathbb{R}^d\times\mathbb{R}^d}\mathrm{e}^{-V_t(x)-V_t(y)-\lambda W_t(x,y)}\mathrm{d}x\mathrm{d}y<\infty, \quad \lambda>0, \quad t\ge 0. \tag{2.9}$$

Consider the following PDE on $\mathscr{D}_2$, the space of all probability density functions on $\mathbb{R}^d$ such that the corresponding probability measure is in $\mathscr{P}_2$:

$$\partial\rho_t=\mathrm{div}\{\nabla\rho_t-\rho_t\nabla(V_t+W_t\circledast\rho_t)\}, \tag{2.10}$$

where for a probability measure μ or a probability density function ρ

$$W_t\circledast\mu:=\int_{\mathbb{R}^d}W_t(\cdot,y)\mu(\mathrm{d}y), \quad W_t\circledast\rho:=\int_{\mathbb{R}^d}W_t(\cdot,y)\rho(y)\mathrm{d}y.$$

We will use $\nabla^{(1)}$ and $\nabla^{(2)}$ to denote the gradient operators in the first and second components on the product space $\mathbb{R}^d\times\mathbb{R}^d$, so that

$$\big\|\nabla^{(1)}\nabla^{(2)}W_t(x,y)\big\|:=\sup_{u,v\in\mathbb{R}^d,|u|,|v|\le 1}\big|\nabla_u^{(1)}\nabla_v^{(2)}W_t(x,y)\big|, \quad t\ge 0, \quad x,y\in\mathbb{R}^d,$$

where ∇_u stands for the directional derivative along u. We let

$$\big\|\nabla^{(1)}\nabla^{(2)}W_t\big\|_\infty:=\sup_{x,y\in\mathbb{R}^d}\big\|\nabla^{(1)}\nabla^{(2)}W_t(x,y)\big\|.$$

For any probability density ρ on $\mathbb{R}^d$ and any $s\ge 0$, let $P^*_{s,t}\rho$ be the solution of (2.10) for $t\ge s$ and $\rho_s=\rho$. If (V_t,W_t) is t_0-periodic, $\bar{\rho}_0\in\mathscr{D}_2$ is called an invariant solution of (2.9) if $P^*_{0,t_0}\bar{\rho}_0=\bar{\rho}_0$. In this case, let

$$\bar{\rho}_s:=P^*_{0,s}\bar{\rho}_0, \quad s\in(0,t_0).$$

Moreover, for any two probability density functions ρ_1,ρ_2,

$$\mathbb{H}(\rho_1|\rho_2):=\mathbb{H}\big(\rho_1(x)\mathrm{d}x|\rho_2(x)\mathrm{d}x\big).$$

Let I_d be the $d\times d$ identity matrix. We have the following consequence of Theorem 2.1, where a simple example for (2.11) to hold is

$$V_t(x)=\frac{\gamma_t}{2}|x|^2,\quad W_t(x,z)=\beta_t|x-z|^2,$$

where γ and β are positive functions on $[0,t_0]$.

Corollary 2.1. *Let (V_t,W_t) be t_0-periodic for some $t_0>0$, and there exists*

$$\gamma\in L_{loc}([0,t_0];\mathbb{R})$$

such that $\lambda:=\int_0^{t_0}\gamma_t\mathrm{d}t>0$ and

$$\mathrm{Hess}_{V_t+W_t(\cdot,z)}\geq\big(\gamma_t+\big\|\nabla^{(1)}\nabla^{(2)}W_t\big\|_\infty\big)I_d,\quad t\in[0,t_0],\quad z\in\mathbb{R}^d. \tag{2.11}$$

Then (2.10) *has a unique invariant solution $\bar\rho_0$ such that*

$$\begin{aligned}&\max\big\{\mathbb{W}_2\big(\rho_{nt_0}(x)\mathrm{d}x,\bar\rho_0(x)\mathrm{d}x\big)^2,\mathbb{H}\big(P^*_{nt_0}\rho|\bar\rho_0\big)\big\}\\ \leq\,& c\mathrm{e}^{-\lambda n}\min\big\{\mathbb{W}_2\big(\rho_0(x)\mathrm{d}x,\bar\rho_0(x)\mathrm{d}x\big)^2,\mathbb{H}(\rho|\bar\rho_0)\big\},\quad n\in\mathbb{N},\quad\rho\in\mathscr{D}_2.\end{aligned} \tag{2.12}$$

2.3 Proofs

We first prove the following lemma which also applies to the non-periodic case.

Lemma 2.1. *Assume $(H_1),(H_2)$. For any $t\geq s\geq 0$, let*

$$\phi_{s,t}:=\lambda(s,t)^2\left(\frac{\kappa_1(s,t)}{1-\mathrm{e}^{-\kappa_1(s,t)}}+\frac{(t-s)\kappa_2(s,t)^2}{2}\mathrm{e}^{2(t-s)\kappa_1(t)+2\kappa_2(t)}\right). \tag{2.13}$$

(1) *For any $\varepsilon>0,c>0$ and $\nu\in T_c$,*

$$\begin{aligned}&\mathbb{H}(P^*_{s,t}\mu|P^*_{s,t}\nu)\\ \leq\,&\phi_{t-\varepsilon,t}\mathrm{e}^{\int_s^{t-\varepsilon}(K_1+K_2)(r)\mathrm{d}r}\mathbb{W}_2(\mu,\nu)^2\\ \leq\,& c\phi_{t-\varepsilon,t}\mathrm{e}^{\int_s^{t-\varepsilon}(K_1+K_2)(r)\mathrm{d}r}\mathbb{H}(\mu,\nu),\quad t\geq s+\varepsilon,\quad s\geq0,\quad \mu\in\mathscr{P}_2.\end{aligned} \tag{2.14}$$

(2) *If $(H_3)(2)$ holds and $\nu\in Log_c$ for some constant $c>0$, then*

$$\begin{aligned}&\mathbb{H}(P^*_{r,t}\mu|P^*_{s,t}\nu)\\&\le\phi_t \mathrm{e}^{\int_r^{t-\varepsilon}(K_1+K_2)(r)\mathrm{d}r}\mathbb{W}_2(\mu,P^*_{s,r}\nu)^2\\&\le c(s,r)\phi_{t-\varepsilon,t}\mathrm{e}^{\int_r^{t-\varepsilon}(K_1+K_2)(r)\mathrm{d}r}\mathbb{H}(\mu|P^*_{s,r}\nu),\quad t\ge r+\varepsilon,\quad r\ge s\ge 0,\quad \mu\in\mathscr{P}_2\end{aligned}\tag{2.15}$$

holds for

$$c(s,r):=cA_r^2\lambda(s,r)^2\mathrm{e}^{-2\int_s^r\gamma_\theta\mathrm{d}\theta}+4A_r^2\int_s^r\mathrm{e}^{-2\int_\tau^r\gamma_\theta\mathrm{d}\theta}\mathrm{d}\tau,\quad r\ge s\ge 0.$$

Proof. (1) By [11, Theorem 4.1] or [7, Theorem 4.1], assumption (H_2) implies

$$\mathbb{H}(P^*_{s,t}\mu|P^*_{s,t}\nu)\le\phi_{s,t}\mathbb{W}_2(\mu,\nu)^2,\quad t\ge s\ge 0,\quad \mu,\nu\in\mathscr{P}_2.$$

So, for $t\ge s+\varepsilon$, we obtain

$$\begin{aligned}\mathbb{H}(P^*_{s,t}\mu|P^*_{s,t}\nu)&=\mathbb{H}(P^*_{t-\varepsilon,t}P^*_{s,t-\varepsilon}\mu|P^*_{t-\varepsilon,t}P^*_{s,t-\varepsilon}\nu)\\&\le\phi_{t-\varepsilon,t}\mathbb{W}_2(P^*_{s,t-\varepsilon}\mu,P^*_{s,t-\varepsilon}\nu)^2.\end{aligned}\tag{2.16}$$

Next, by [11, Theorem 3.1], assumption (H_1) implies

$$\mathbb{W}_2(P^*_{s,t-\varepsilon}\mu,P^*_{s,t-\varepsilon}\nu)^2\le\mathrm{e}^{\int_s^{t-\varepsilon}(K_1(r)+K_2(r))\mathrm{d}r}\mathbb{W}_2(\mu,\nu)^2.$$

Combining this with (2.16) and applying (2.4) we prove (2.14).

(2) Noting that $P^*_{s,t}\nu=P^*_{r,t}(P^*_{s,r}\nu)$, to deduce (2.15) from (2.14) we need only to prove $P^*_{s,r}\nu\in T_{c(s,r)}$ which follows from $P^*_{s,r}\nu\in Log_{c(s,r)}$. To this end, we let $\nu_t:=P^*_{s,t}\nu$ and consider the decoupled (classical) SDE of (1.3)

$$\mathrm{d}X_t^\nu=b_t\big(X_t^\nu,\nu_t\big)\mathrm{d}t+\sigma_t\big(X_t^\nu\big)\mathrm{d}W_t,\quad t\ge s,\quad X_0^\nu\in\mathbb{R}^d.\tag{2.17}$$

For any $\mu\in\mathscr{P}$, let $(P^\nu_{s,t})^*\mu=\mathscr{L}_{X^\nu_{s,t}}$ for $\mathscr{L}_{X^\nu_s}=\mu$. Then

$$P^*_{s,r}\nu=\big(P^\nu_{s,t}\big)^*\nu,\quad r\ge s.\tag{2.18}$$

Now, for $\nu\in Log_c$, $\|\sigma_s^{-1}\|\le\lambda_s$ implies

$$\nu(f\log f)\le\frac{c\lambda_s^2}{4}\nu\left(\frac{|\sigma_s^*\nabla f|^2}{f}\right),\quad 0<f\in C_b^1(\mathbb{R}^d),\quad \nu(f)=1.\tag{2.19}$$

According to [5, Theorem 4.1] for the time inhomogeneous Markov semigroup associated with (2.17), we remark that in this result $\Gamma(f)$ is misprint from $\Gamma(f)/f$ (see Lemma 5.1 below for $D=\mathbb{R}^d$), (H_3) and (2.17) yield that $\nu_r:=P_{s,r}^*\nu=(P_{s,t}^\nu)^*\nu$ satisfies

$$\nu_r(f\log f)\le\frac{c(s,r)}{4A_r^2}\nu_r\left(\frac{|\sigma_r^*\nabla f|^2}{f}\right),\quad 0<f\in C_b^1(\mathbb{R}^d),\quad \nu_r(f)=1. \tag{2.20}$$

Since $\|\sigma_r\|_\infty\le A_r$, this implies $P_{s,r}^*\nu\in T_{c(s,r)}$ as desired. □

Proof of Theorem 2.1. By shifting a time $s\in[0,t_0)$, for simplicity, we assume $s=0$.

(1) By (2.6) and the t_0-periodicity, the uniqueness of $\bar{\mu}$ and (2.7) follows from (2.1). So, it suffices to prove the existence of $\bar{\mu}_0$.

Take

$$\mu_n:=P_{0,nt_0}^*\delta_0,\quad n\in\mathbb{N}. \tag{2.21}$$

We intend to prove that μ_n converges to some $\bar{\mu}_0\in\mathscr{P}_2$ as $n\to\infty$, so that by a standard argument the semigroup property of $\bar{P}_n^*:=P_{0,nt_0}^*$

$$\bar{P}_{n+m}^*=\bar{P}_n^*\bar{P}_m^*,\quad n,m\in\mathbb{Z}_+,$$

we conclude that $\bar{\mu}_0$ is an invariant probability measure. To this end, it remains to show that $\{\mu_n\}_{n\ge1}$ is a $\mathbb{W}_2$-Cauchy sequence, i.e.

$$\lim_{n\to\infty}\sup_{k\ge1}\mathbb{W}_2(\mu_n,\mu_{n+k})=0. \tag{2.22}$$

By (2.1), (1.4) and (2.6), we obtain

$$\mathbb{W}_2(\mu_n,\mu_{n+k})^2\le e^{-\int_0^{nt_0}(K_1(r)+K_2(r))dr}\mathbb{W}_2\big(\delta_0,P_{0,kt_0}^*\delta_0\big)^2=e^{-\lambda n}\mathbb{E}|X_{kt_0}|^2, \tag{2.23}$$

where X_t solves (1.3) with $X_0=0$. By taking $y=0,\nu=\delta_0$ in (H_1), and noting that the periodicity together with (H_1) implies that $|b_t(0,\delta_0)|+\|\sigma_t(0)\|$ is bounded and $\|\sigma_t(x)\|\le c_0(1+|x|)$ for some constant $c_0>0$, we find constants $c_1,c_2>0$ such that

$$\begin{aligned}
&2\langle b_t(x,\mu),x\rangle+\|\sigma_t(x)\|_{HS}^2\\
&=2\langle b_t(x,\mu)-b_t(0,\delta_0),x-0\rangle+\|\sigma_t(x)-\sigma_t(0)\|_{HS}^2\\
&\quad+2\langle b_t(0,\delta_0),x\rangle-\|\sigma_t(0)\|_{HS}^2+2\langle\sigma_t(x),\sigma_t(0)\rangle_{HS}\\
&\le K_1(t)|x|^2+K_2(t)\mu(|\cdot|^2)+c_1(1+|x|)\\
&\le c_2+\left(K_1(t)+\frac{\lambda}{2t_0}\right)|x|^2+K_2(t)\mu(|\cdot|^2),\quad t\ge0,\quad x\in\mathbb{R}^d,\quad \mu\in\mathscr{P}_2.
\end{aligned}$$

So, by applying Itô's formula to (1.3) for $X_0=0$, we obtain

$$\mathrm{d}|X_t|^2 \le \left\{c_2+\left(K_1(t)+\frac{\lambda}{2t_0}\right)|X_t|^2+K_2(t)\mathbb{E}|X_t|^2\right\}\mathrm{d}t+\mathrm{d}M_t$$

for some martingale M_t. By Duhamel's formula, this and $X_0=0$ implies

$$\mathbb{E}|X_t|^2 \le c_2\int_0^t \mathrm{e}^{\int_s^t (K_1(r)+K_2(r)+\frac{\lambda}{2t_0})\mathrm{d}r}\mathrm{d}s, \quad t\ge 0. \tag{2.24}$$

By (2.6) and the t_0-periodicity, we find a constant $C>0$ such that

$$\int_s^{s+kt_0}\left(K_1(r)+K_2(r)+\frac{\lambda}{2t_0}\right)\mathrm{d}r=-\frac{\lambda k}{2}<0, \quad s\ge 0, \quad k\in\mathbb{Z}_+.$$

So, letting $\lfloor r\rfloor:=\sup\{n\in\mathbb{Z}_+: r\ge n\}$ for $r\ge 0$, (2.24) implies

$$\begin{aligned}\sup_{t\ge 0}\mathbb{E}|X_t|^2 &\le \sup_{t\ge 0} c_2\int_0^t \mathrm{e}^{-\frac{\lambda\lfloor (t-s)/t_0\rfloor}{2}+\int_0^{t_0}|K_1(r)+K_2(r)+\frac{\lambda}{2t_0}|\mathrm{d}r}\mathrm{d}s\\ &\le C\int_0^t \mathrm{e}^{-\frac{(t-s)\lambda}{2}}\mathrm{d}s \le \frac{2C}{\lambda}<\infty. \end{aligned}\tag{2.25}$$

Combining this with (2.23), we prove the desired (2.22).

(2) By (2.15) and (2.7), it suffices to find a constant $c>0$ such that $\bar\mu_0\in Log_c$.

a) When $(H_3)(1)$ holds, let $(\bar P_{s,t})_{t\ge s}$ be the semigroup associated with the SDE

$$\mathrm{d}\bar X_t=b_t(\bar X_t,\bar\mu_0)\mathrm{d}t+\sigma_t\mathrm{d}W_t, \tag{2.26}$$

that is, letting $(\bar X_{s,t}^x)_{t\ge s}$ be the solution starting from x at time s,

$$\bar P_{s,t}f(x):=\mathbb{E}f\big(\bar X_{s,t}^x\big), \quad f\in\mathbf{B}_b(\mathbb{R}^d), \quad t\ge 0.$$

By (H_1) which implies $K_2\ge 0$, we have

$$\begin{aligned}&2\langle b_t(x,\bar\mu_0)-b_t(y,\bar\mu_0),x-y\rangle \le K_1(t)|x-y|^2, \quad x,y\in\mathbb{R}^d, \quad t\ge 0,\\ &\int_0^{t_0}K_1(s)\mathrm{d}s\le -\lambda<0.\end{aligned}\tag{2.27}$$

Noting that σ_t is constant, this implies

$$\big|\bar X_{s,t}^x-\bar X_{s,t}^y\big|\le \mathrm{e}^{\frac{1}{2}\int_s^t K_1(r)\mathrm{d}r}|x-y|, \quad x,y\in\mathbb{R}^d,$$

so that for any $f\in C_b^1(\mathbb{R}^d)$,

$$|\nabla \bar{P}_{s,t}f|\le \mathrm{e}^{\frac{1}{2}\int_s^t K_1(s)\mathrm{d}s}\bar{P}_{s,t}|\nabla f|\le c_1\mathrm{e}^{-\frac{\lambda}{2}\lfloor\frac{t-s}{t_0}\rfloor}\bar{P}_t|\nabla f|,\quad t\ge s\ge 0 \tag{2.28}$$

for some constant $c_1>0$. So, for any $f\in C_b^1(\mathbb{R}^d)$,

$$\begin{aligned}&\bar{P}_t(f^2\log f^2)-\big(\bar{P}_tf^2\big)\log\big(\bar{P}_tf^2\big)\\&=\int_0^t\frac{\mathrm{d}}{\mathrm{d}s}\bar{P}_s\big\{\big(\bar{P}_{s,t}f^2\big)\log\big(\bar{P}_{s,t}f^2\big)\big\}\mathrm{d}s\\&=\int_0^t\bar{P}_s\frac{\big|\sigma_s^*\nabla\bar{P}_{s,t}f^2\big|^2}{\bar{P}_{s,t}f^2}\mathrm{d}s\le c_1^2\int_0^t\|\sigma_s\|^2\mathrm{e}^{-\lambda\lfloor\frac{t-s}{t_0}\rfloor}\bar{P}_s\bar{P}_{s,t}|\nabla f|^2\mathrm{d}s\\&=\big(\bar{P}_t|\nabla f|^2\big)c_1^2\int_0^t\|\sigma_s\|^2\mathrm{e}^{-\lambda\lfloor\frac{t-s}{t_0}\rfloor}\mathrm{d}s,\quad t\ge 0.\end{aligned}$$

By the t_0-periodicity and $\|\sigma_\cdot\|^2\in L^1([0,t_0])$, we obtain

$$\begin{aligned}\int_0^t\|\sigma_s\|^2\mathrm{e}^{-\lambda\lfloor\frac{t-s}{t}{}_0\rfloor}\mathrm{d}s&\le\sum_{i=0}^{\lfloor t/t_0\rfloor}\int_{it_0}^{(i+1)t_0}\|\sigma_s\|^2\mathrm{e}^{-\lambda(\lfloor\frac{t}{t_0}\rfloor-i-1)}\mathrm{d}s\\&\le\left(\int_0^{t_0}\|\sigma_s\|^2\mathrm{d}s\right)\sum_{i=0}^{\infty}\mathrm{e}^{-(i-1)\lambda}=:c<\infty,\quad t\ge 0,\end{aligned}$$

so that there exists a constant $c>0$ such that

$$\bar{P}_t(f^2\log f^2)-\big(\bar{P}_tf^2\big)\log\big(\bar{P}_tf^2\big)\le c\bar{P}_t|\nabla f|^2,\quad t\ge 0,\quad f\in C_b^1(\mathbb{R}^d).$$

Moreover, by (2.27), (2.26) is exponential ergodic with unique invariant probability measure $\bar{\mu}_0$ as it reduces to (1.3) when $\mathscr{L}_{\bar{X}_0}=\bar{\mu}_0$. By taking $t=nt_0$ and letting $n\to\infty$, we prove $\mu_n\in Log_c$ for all $n\ge 1$.

b) When $(H_3)(2)$ holds with $\gamma:=\int_0^{t_0}\gamma_s\mathrm{d}s>0$, we apply Lemma 2.1 for $s=0$ and $\nu=\delta_0$. Then (2.19) holds for $c=0$, so that by the t_0-periodic and $\gamma>0$, we find a constant $c'>0$ such that

$$c(0,nt_0)=4\lambda(t_0)^2\int_0^{nt_0}\mathrm{e}^{-2\int_\tau^{nt_0}\gamma_\theta\mathrm{d}\theta}\mathrm{d}\tau\le c',\quad n\in\mathbb{N},$$

where $\lambda(t_0)$ is in (H_2). Moreover, by (2.20), $\mu_n:=P_{0,nt_0}^*\delta_0$ satisfies

$$\begin{aligned}\mu_n(f^2\log f^2)&\le\frac{c(0,nt_0)}{\|\sigma_{t_0}\|_\infty^2}\mu_n\big(\big|\sigma_{t_0}^*\nabla f\big|^2\big)\\&\le c(0,nt_0)\mu_n(|\nabla f|^2),\quad 0<f\in C_b^1(\mathbb{R}^d),\quad \mu_n(f^2)=1.\end{aligned}$$

Therefore, $\mu_n\in Log_{c'}$ for all $n\ge 1$, which together with (2.7) implies $\bar{\mu}_0\in Log_{c'}$. □

Proof of Corollary 2.1. It is easy to see that for any $s\geq 0$ and probability density function ρ, $P_{s,t}^*\rho$ is the density function of $\mathscr{L}_{X_t}$ for X_t solving (1.3) from time s with $\mathscr{L}_{X_s}=\rho(x)\mathrm{d}x$ and

$$\sigma_t(x):=\sqrt{2}I_d,\quad b_t(x,\mu):=-\nabla\{V_t+W_t\circledast\mu\}(x),\quad t\geq 0,\quad x\in\mathbb{R}^d,\quad \mu\in\mathscr{P}_2. \tag{2.29}$$

Then (2.11) implies

$$\begin{aligned}
&2\langle b_t(x,\mu)-b_t(y,\nu),x-y\rangle\\
&=2\langle b_t(x,\mu)-b_t(y,\mu),x-y\rangle+2\langle b_t(y,\mu)-b_t(y,\nu),x-y\rangle\\
&=-2\int_{\mathbb{R}^d}\mu(\mathrm{d}z)\int_0^1\langle \mathrm{Hess}_{V_t+W_t(\cdot,z)}\big(x+r(y-x)\big)(x-y),x-y\rangle\mathrm{d}r\\
&\quad+2\langle\nu\big(\nabla^{(1)}W_t(y,\cdot)\big)-\mu\big(\nabla^{(1)}W_t(y,\cdot)\big),x-y\rangle\\
&\leq-2\big(\gamma_t+\|\nabla^{(1)}\nabla^{(2)}W_1\|_\infty\big)|x-y|^2+2\|\nabla^{(1)}\nabla^{(2)}W_t\|_\infty|x-y|\mathbb{W}_1(\mu,\nu)\\
&\leq-2\big(\gamma_t+\|\nabla^{(1)}\nabla^{(2)}W_1\|_\infty\big)|x-y|^2+2\|\nabla^{(1)}\nabla^{(2)}W_t\|_\infty|x-y|\mathbb{W}_2(\mu,\nu).
\end{aligned}$$

Thus, (H_1) holds for

$$K_1(t)=-2\gamma_t-\|\nabla^{(1)}\nabla^{(2)}W_t\|_\infty,\quad K_2(t)=\|\nabla^{(1)}\nabla^{(2)}W_t\|_\infty,$$

and (H_2) holds for

$$\kappa_1(t)=0,\quad \kappa_2(t)=2\sup_{s\in[0,t]}\|\nabla^{(1)}\nabla^{(2)}W_s\|_\infty.$$

Moreover, since σ is constant, $(H_3)(1)$ holds. Therefore, this result follows from Theorem 2.1. □

3 Ergodicity for partially dissipative models

For any

$$\psi\in\Psi=\big\{\psi\in C^2\big([0,\infty)\big):\psi(0)=0,\psi'>0,\|\psi'\|_\infty<\infty\big\},$$

let

$$\begin{aligned}
&\mathscr{P}_\psi:=\{\mu\in\mathscr{P}:\mu(\psi(|\cdot|))<\infty\},\\
&\mathbb{W}_\psi(\mu,\nu):=\inf_{\pi\in\mathscr{C}(\mu,\nu)}\int_{\mathbb{R}^d\times\mathbb{R}^d}\psi(|x-y|)\pi(\mathrm{d}x,\mathrm{d}y),\quad \mu,\nu\in\mathscr{P}_\psi.
\end{aligned}$$

Then $\mathscr{P}_\psi$ is complete under the quasi-metric $\mathbb{W}_\psi$, and when ψ is concave, $\mathbb{W}_\psi$ is a metric, see for instance [10]. Let $\|\cdot\|_{Lip}$ be the Lipschitz constant for functions on $\mathbb{R}^d$. We assume

(H_4) (Ellipticity) $\sigma_t(x,\mu)=\sigma_t(x)$ does not depend on μ, and there exist

$$\alpha\in L^1_{loc}\big([0,\infty);(0,\infty)\big)$$

and a measurable map

$$\hat\sigma:[0,\infty)\times\mathbb{R}^d \ \to\ \mathbb{R}^d\otimes\mathbb{R}^d$$

such that

$$\begin{aligned}&\sup_{t\in[0,T]}\{\|\sigma_t\|_{Lip}+\|\hat\sigma_t\|_{Lip}\}<\infty, \qquad T>0,\\&\sigma_t(x)\sigma_t(x)^*=\alpha_t I_d+\hat\sigma_t(x)\hat\sigma_t(x)^*,\quad t\ge0,\quad x\in\mathbb{R}^d.\end{aligned}$$

(H_5) (Partial Dissipativity) Let $\psi\in\Psi,\gamma\in C([0,\infty)\times[0,\infty))$ with $\gamma_t(r)\le Kr$ for some constant $K>0$ and all $r\ge0$, such that

$$2\alpha_t\psi''(r)+(\gamma_t\psi')(r)\le-\kappa_t\psi(r),\quad r\ge0 \tag{3.1}$$

holds for some $\kappa\in L^1_{loc}([0,\infty);\mathbb{R})$. Moreover, b is bounded on bounded subsets of $[0,\infty)\times\mathbb{R}^d\times\mathscr{P}_\psi$, and there exists $\theta\in L^1_{loc}([0,\infty);(0,\infty))$ such that

$$\begin{aligned}&\langle b_t(x,\mu)-b_t(y,\nu),x-y\rangle+\frac12\|\hat\sigma_t(x)-\hat\sigma_t(y)\|_{HS}^2\\&\le|x-y|\big\{\theta_t\mathbb{W}_\psi(\mu,\nu)+\gamma_t(|x-y|)\big\},\quad t\ge0,\quad x,y\in\mathbb{R}^d,\quad \mu,\nu\in\mathscr{P}_\psi.\end{aligned}\tag{3.2}$$

Theorem 3.1. *Assume (H_4) and (H_5), with $\psi''\le0$ if $\hat\sigma_t$ is non-constant for some $t\ge0$. Then* (1.3) *is well-posed with distributions in $\mathscr{P}_\psi$, and P_t^* satisfies*

$$\mathbb{W}_\psi(P_t^*\mu,P_t^*\nu)\le \mathrm{e}^{-\int_0^t\{\kappa_s-\theta_s\|\psi'\|_\infty\}\mathrm{d}s}\mathbb{W}_\psi(\mu,\nu),\quad t\ge0,\quad \mu,\nu\in\mathscr{P}_\psi.\tag{3.3}$$

Consequently, if (b_t,σ_t) is t_0-periodic, $\psi'(t)\le C\psi'(s)$ for some constant $C>1$ and all $t\ge s\ge0$, and

$$\lambda:=\int_0^{t_0}\{\kappa_s-\theta_s\|\psi'\|_\infty\}\mathrm{d}s>0,$$

then (1.3) *has a unique invariant probability measure $\bar\mu_0\in\mathscr{P}_\psi$ such that*

$$\mathbb{W}_\psi(P^*_{s,s+nt_0}\mu,\bar\mu_0)\le\mathrm{e}^{-n\lambda}\mathbb{W}_\psi(\mu,\bar\mu_0),\quad n\in\mathbb{N},\quad \mu\in\mathscr{P}_\psi.\tag{3.4}$$

Proof. The well-posedness and (3.3) follow from [12, Theorem 3.1] by using coupling methods. So, it suffices to prove the existence of the invariant probability measure $\bar{\mu}_0$ when $\lambda>0$ and the coefficients are t_0-periodic. Let $x_0\in\mathbb{R}^d$. It suffices to show that the sequence $\{P^*_{nt_0}\delta_{x_0}\}_{n\ge1}$ is a $\mathbb{W}_\psi$-Cauchy sequence so that its limit is an invariant probability measure of (1.3). By (3.3), we have

$$\mathbb{W}_\psi\big(P^*_{nt_0}\delta_{x_0},P^*_{(n+m)t_0}\delta_{x_0}\big)\le C\mathrm{e}^{-n\lambda}\mathbb{W}_\psi\big(\delta_{x_0},P^*_{mt_0}\delta_{x_0}\big),\quad n,m\ge1.$$

Since $\lambda>0$, it suffices to prove

$$\sup_{m\ge1}\mathbb{W}_\psi\big(\delta_{x_0},P^*_{mt_0}\delta_{x_0}\big)<\infty. \tag{3.5}$$

By $\psi'(t)\le C\psi'(s)$ for $t\ge s$, we have

$$\psi(s+t)-\psi(s)=\int_s^{s+t}\psi'(r)\mathrm{d}r\le C\int_0^t\psi'(r)\mathrm{d}r=C\psi(t),\quad s,t\ge0.$$

This implies

$$\psi\left(\sum_{i=1}^n s_i\right)\le C\sum_{i=1}^n\psi(s_i),\quad s_i\ge0,\quad n\ge1.$$

Consequently, by (3.3) and $\lambda>0$, we obtain

$$\begin{aligned}\mathbb{W}_\psi\big(\delta_{x_0},P^*_{nt_0}\delta_{x_0}\big)&\le C\sum_{i=0}^{n-1}\mathbb{W}_\psi\big(P^*_{it_0}\delta_{x_0},P^*_{(i+1)t_0}\delta_{x_0}\big)\\&\le C\mathbb{W}_\psi\big(\delta_{x_0},P^*_{t_0}\delta_{x_0}\big)\sum_{i=0}^\infty\mathrm{e}^{-i\lambda}<\infty.\end{aligned}$$

Therefore, (3.5) holds. □

To illustrate Theorem 3.1, we present below an example associated with time-inhomogeneous granular media equations. Let $\mathbb{W}_1=\mathbb{W}_\psi$ and $\mathscr{P}_1(\mathbb{R}^d)=\mathscr{P}_\psi(\mathbb{R}^d)$ for $\psi(r)=r$.

Example 3.1. Let $\alpha\in L^1([0,t_0]:(0,\infty))$ and

$$V:[0,t_0]\times\mathbb{R}^d\ \to\ \mathbb{R},\quad W:[0,t_0]\times\mathbb{R}^d\times\mathbb{R}^d\ \to\ \mathbb{R}$$

be measurable with $V_t\in C^2(\mathbb{R}^d)$, $W_t\in C^2(\mathbb{R}^d\times\mathbb{R}^d)$, and for some constants $R,\theta_1,\theta_2>0$,

$$\mathrm{Hess}_{V_t+W_t(\cdot,z)}\ge\big(\theta_2\alpha_t1_{\{|\cdot|>R/2\}}-\theta_1\alpha_t1_{\{|\cdot|\le R/2\}}\big)I_d,\quad t\in[0,t_0],\quad z\in\mathbb{R}^d. \tag{3.6}$$

Consider (1.3) with t_0-periodic coefficients

$$\sigma_t=\sqrt{\alpha_t}I_d,\quad b_t(x,\mu):=-\nabla\{V_t+W_t\circledast\mu\}(x),\quad (t,x,\mu)\in[0,t_0]\times\mathbb{R}^d\times\mathscr{P}_1. \tag{3.7}$$

Let $\gamma(r)=\theta_1(r\wedge R)-\theta_2(r-R)^+$ for $r\geq 0$, and

$$\psi(r):=\int_0^r \mathrm{e}^{-\int_0^s\gamma(u)\mathrm{d}u}\mathrm{d}s\int_s^\infty t\mathrm{e}^{\int_0^t\gamma(u)\mathrm{d}u}\mathrm{d}t,\quad r\geq 0.$$

Then

$$c_1(\psi):=\inf_{r\geq 0}\psi'(r)>0,\quad c_2(\psi):=\sup_{r\geq 0}\psi'(r)<\infty.$$

If

$$\lambda:=2\int_0^{t_0}\left(\frac{\alpha_t}{c_2(\psi)}-\frac{\|\nabla^{(1)}\nabla^{(2)}W_t\|_\infty}{c_1(\psi)}\right)\mathrm{d}t>0,$$

then (1.3) has a unique invariant probability measure $\bar{\mu}_0\in\mathscr{P}_1$ such that

$$\mathbb{W}_\psi\big(P_{nt_0}^*\mu,\bar{\mu}_0\big)\leq \mathrm{e}^{-\lambda n}\mathbb{W}_\psi(\mu,\bar{\mu}_0),\quad n\in\mathbb{N},\quad \mu\in\mathscr{P}_1=\mathscr{P}_\psi.$$

Consequently,

$$\mathbb{W}_1\big(P_{nt_0}^*\mu,\bar{\mu}_0\big)\leq \frac{c_2(\psi)}{c_1(\psi)}\mathrm{e}^{-\lambda n}\mathbb{W}_1(\mu,\bar{\mu}_0),\quad n\in\mathbb{N},\quad \mu\in\mathscr{P}_1.$$

Proof. It is easy to see that $\psi\in C^2([0,\infty)$ with $\psi'>0$ and

$$\lim_{r\to\infty}\psi'(r)=\lim_{r\to\infty}\frac{\int_r^\infty t\mathrm{e}^{\int_0^t\gamma(u)\mathrm{d}u}\mathrm{d}t}{\mathrm{e}^{\int_0^r\gamma(u)\mathrm{d}u}}=\lim_{r\to\infty}\frac{r}{-\gamma(r)}=\frac{1}{\theta_2}.$$

So, $0<c_1(\psi)<c_2(\psi)<\infty$ and

$$c_1(\psi)\mathbb{W}_1\leq\mathbb{W}_\psi\leq c_2(\psi)\mathbb{W}_1.$$

By Theorem 3.1, it suffices to verify (3.1) and (3.2) for

$$\gamma_t(r):=2\alpha_t\gamma(r),\quad \theta_t:=\frac{2}{c_1(\psi)}\|\nabla^{(1)}\nabla^{(2)}W_t\|_\infty,\quad \kappa_t:=\frac{2\alpha_t}{c_2(\psi)}. \tag{3.8}$$

Firstly, by the definitions of γ and $\psi,\gamma_t:=2\alpha_t\gamma$ in (3.8), we have

$$2\alpha_t\psi''(r)+2\alpha_t(\psi'\gamma)(r)=-2\alpha_t r\leq -\frac{2\alpha_t}{c_2(\psi)}\psi(r),\quad r\geq 0.$$

Then (3.1) holds for γ_t and κ_t in (3.8).

Next, by (3.6) and (3.7), we have $\hat{\sigma}=0$, and as in the proof of Corollary 2.1,

$$\begin{aligned}
&2\langle b_t(x,\mu)-b_t(y,\nu),x-y\rangle \\
&=-2\int_{\mathbb{R}^d}\mu(\mathrm{d}z)\int_0^1\big\langle \mathrm{Hess}_{V_t+W_t(\cdot,z)}\big(x+r(x-y)\big)(x-y),x-y\big\rangle \mathrm{d}r \\
&\quad +2\big\langle \nu\big(\nabla^{(1)}W_t(y,\cdot)\big)-\mu\big(\nabla^{(1)}W_t(y,\cdot)\big),x-y\big\rangle \\
&\le 2|x-y|^2\int_0^1\big(\theta_1\alpha_t 1_{\{|x+r(y-x)|\le R/2\}}-\theta_2\alpha_t 1_{\{|x+r(y-x)|\ge R/2\}}\big)\mathrm{d}r \\
&\quad +2\big\|\nabla^{(1)}\nabla^{(2)}W_t\big\|_\infty \mathbb{W}_1(\mu,\nu)|x-y| \\
&\le 2\alpha_t|x-y|\gamma(|x-y|)+\frac{2}{c_1(\psi)}\big\|\nabla^{(1)}\nabla^{(2)}W_t\big\|_\infty|x-y|\mathbb{W}_\psi(\mu,\nu)
\end{aligned}$$

holds for any $t\in[0,t_0]$, $x,y\in\mathbb{R}^d$ and $\mu,\nu\in\mathscr{P}_\psi=\mathscr{P}_1$. Hence, (3.2) holds for γ_t and θ_t in (3.8). □

4 Ergodicity for non-dissipative models

We consider the fully non-dissipative case such that [12, Theorem 2.1] is extended to the periodic setting. For any $t\ge 0$ and $\mu\in\mathscr{P}$, consider the second-order differential operator

$$L_{t,\mu}:=\frac{1}{2}\mathrm{tr}\big\{\sigma_t\sigma_t^*\nabla^2\big\}+b_t(\cdot,\mu)\cdot\nabla. \tag{4.1}$$

For any positive measurable function V on $\mathbb{R}^d$, let

$$\mathscr{P}_V:=\{\mu\in\mathscr{P}:\mu(V)<\infty\}.$$

(H_7) (Lyapunov Condition) There exist $0\le V\in C^2(\mathbb{R}^d)$ with $\lim_{|x|\to\infty}V(x)=\infty$ and $K_0,K_1\in L^1_{loc}([0,\infty);\mathbb{R})$ such that

$$\sup_{t\ge 0;x\in\mathbb{R}^d}\frac{|\sigma(t,x)\nabla V(x)|}{1+V(x)}<\infty, \tag{4.2}$$

$$L_{t,\mu}V\le K_0(t)-K_1(t)V,\quad t\ge 0,\quad \mu\in\mathscr{P}_V. \tag{4.3}$$

Since $\lim_{|x|\to\infty}V(x)=\infty$, (4.3) controls the long distance behaviour of the associated stochastic system. To ensure the exponential ergodicity, we also need conditions in short distance. For any $l>0$, consider the class

$$\Psi_l:=\big\{\psi\in C^2\big([0,l];[0,\infty)\big):\psi(0)=\psi'(l)=0,\psi'|_{[0,l)}>0\big\}.$$

For each $\psi\in\Psi_l$, we extend it to the half line by setting $\psi(r)=\psi(r\wedge l)$, so that ψ' is non-negative and Lipschitz continuous with compact support and

$$c_\psi:=\sup_{r>0}\frac{r\psi'(r)}{\psi(r)}<\infty. \tag{4.4}$$

For any constant $\beta>0$, define the quasi-distance on $\mathscr{P}_V(\mathbb{R}^d)$

$$\begin{aligned}&\mathbb{W}_{\psi,\beta V}(\mu,\nu)\\ &:=\inf_{\pi\in\mathscr{C}(\mu,\nu)}\int_{\mathbb{R}^d\times\mathbb{R}^d}\psi(|x-y|)\big(1+\beta V(x)+\beta V(y)\big)\pi(\mathrm{d}x,\mathrm{d}y),\quad \mu,\nu\in\mathscr{P}_V.\end{aligned}$$

To prove the exponential convergence of P_t^* under $\mathbb{W}_{\psi,\beta V}$, the dependence on distribution for the drift will be characterized by

$$\begin{aligned}\hat{\mathbb{W}}_{\psi,\beta V}(\mu,\nu)&:=\inf_{\pi\in\mathscr{C}(\mu,\nu)}\frac{\int_{\mathbb{R}^d\times\mathbb{R}^d}\psi(|x-y|)\big(1+\beta V(x)+\beta V(y)\big)\pi(\mathrm{d}x,\mathrm{d}y)}{\int_{\mathbb{R}^d\times\mathbb{R}^d}\psi'(|x-y|)\big(1+\beta V(x)+\beta V(y)\big)\pi(\mathrm{d}x,\mathrm{d}y)}\\ &\geq\frac{\mathbb{W}_{\psi,\beta V}(\mu,\nu)}{\|\psi'\|_\infty\big(1+\beta\mu(V)+\beta\nu(V)\big)},\quad \mu,\nu\in\mathscr{P}_V.\end{aligned} \tag{4.5}$$

(H_8) (Local Monotonicity) b is bounded on bounded subsets of $[0,\infty)\times\mathbb{R}^d\times\mathscr{P}_V$. Moreover, there exist $l>0,\psi\in\Psi_l$ and $u_l,K,\theta\in L^1_{loc}([0,\infty);[0,\infty))$ such that

$$\begin{aligned}&2\alpha_t\psi''(r)+K(t)\psi'(r)\leq -u_l(t)\psi(r),\quad r\in[0,l],\quad t\geq 0,\\ &\langle b_t(x,\mu)-b_t(y,\nu),x-y\rangle+\frac{1}{2}\|\hat\sigma_t(x)-\hat\sigma_t(y)\|_{HS}^2\\ &\leq K_t|x-y|^2+\theta_t|x-y|\hat{\mathbb{W}}_{\psi,\beta V}(\mu,\nu),\quad x,y\in\mathbb{R}^d,\quad \mu,\nu\in\mathscr{P}_V,\quad t\geq 0.\end{aligned}$$

By (H_7), for any $l>0$, we have

$$\kappa_{l,\beta}(t):=\inf_{|x-y|>l}\frac{K_1(t)V(x)+K_1(t)V(y)-2K_0(t)}{\beta^{-1}+V(x)+V(y)}\in\mathbb{R}, \tag{4.6}$$

and when $K_1(t)>0$ and $l>0$ is large enough, $\kappa_{l,\beta}(t)>0$. Moreover, (H_4) and (H_7) imply

$$\begin{aligned}\alpha_{l,\beta}(t):=C_\psi\sup_{|x-y|\in(0,l)}\Big\{&\alpha_t\frac{|\nabla V(x)-\nabla V(y)|}{|x-y|\{\beta^{-1}+V(x)+V(y)\}}\\ &+\frac{\big|\{\hat\sigma_t(x)-\hat\sigma_t(y)\}\big[\big(\hat\sigma_t(\cdot)^*\nabla V\big)(x)+\big(\hat\sigma_t(\cdot)^*\nabla V\big)(y)\big]\big|}{|x-y|\{\beta^{-1}+V(x)+V(y)\}}\Big\}<\infty.\end{aligned} \tag{4.7}$$

For $K_0,\kappa_{l,\beta},\alpha_{l,\beta}$ and u_l given in (H_7), (H_8), (4.6) and (4.7), let

$$\lambda_{l,\beta}(t):=\min\big\{\kappa_{l,\beta}(t),u_l(t)-2K_0(t)\beta-\alpha_{l,\beta}(t)\big\},\quad t\geq 0. \tag{4.8}$$

Theorem 4.1. *Assume (H_4), (H_7) and (H_8) with $\psi''\leq 0$ when $\hat{\sigma}_t(\cdot)$ is non-constant. Then* (1.3) *is well-posed for distributions in $\mathscr{P}_V$, and P_t^* satisfies*

$$\mathbb{W}_{\psi,\beta V}(P_t^*\mu,P_t^*\nu)\leq \mathrm{e}^{-\int_0^t\{\lambda_{l,\beta}(s)-\theta_s\}\mathrm{d}s}\mathbb{W}_{\psi,\beta V}(\mu,\nu),\quad t\geq 0,\quad \mu,\nu\in\mathscr{P}_V. \tag{4.9}$$

Consequently, if (σ_t,b_t) is t_0-periodic and

$$\lambda:=\int_0^{t_0}\{\lambda_{l,\beta}(s)-\theta_s\}\mathrm{d}s>0,\quad \int_0^{t_0}K_1(t)\mathrm{d}t>0,$$

then (1.3) *has a unique invariant probability measure $\bar{\mu}_0\in\mathscr{P}_V$ such that*

$$\mathbb{W}_{\psi,\beta V}\big(P_{nt_0}^*\mu,\bar{\mu}_0\big)\leq \mathrm{e}^{-\lambda n}\mathbb{W}_{\psi,\beta V}(\mu,\bar{\mu}_0),\quad n\in\mathbb{N},\quad \mu\in\mathscr{P}_V. \tag{4.10}$$

Proof. The well-posedness and (4.9) is included in [12, Theorem 2.1]. So, it suffices to prove the existence of invariant probability measure $\bar{\mu}_0\in\mathscr{P}_V$ for the t_0-periodic case with $\lambda>0$. Let $x_0\in\mathbb{R}^d$. By (4.9) we have

$$\mathbb{W}_{\psi,\beta V}\big(P_{nt_0}^*\delta_{x_0},P_{(n+m)t_0}^*\delta_{x_0}\big)\leq \mathrm{e}^{-\lambda n}\mathbb{W}_{\psi,\beta V}\big(\delta_{x_0},P_{mt_0}^*\delta_{x_0}\big),\quad n,m\geq 1.$$

Therefore, it suffices to prove

$$\sup_{m\geq 1}\mathbb{E}V\big(X_{mt_0}\big)<\infty,\quad X_0=x_0, \tag{4.11}$$

which together with the above inequality implies that $\{P_{nt_0}^*\delta_{x_0}\}_{n\geq 1}$ is a $\mathbb{W}_{\psi,\beta V}$-Cauchy sequence and its limit is an invariant probability measure in $\mathscr{P}_V$. By (4.3), Itô's formula and

$$\int_m^{(n+m)t_0}K_1(s)\mathrm{d}s=n\int_0^{t_0}K_1(s)\mathrm{d}s=:n\lambda_0>0,\quad n,m\in\mathbb{N},$$

we obtain

$$\begin{aligned}\mathbb{E}V\big(X_{nt_0}\big)&\leq V(x_0)\mathrm{e}^{-\int_0^{nt_0}K_1(s)\mathrm{d}s}+\int_0^{nt_0}|K_0(s)|\mathrm{e}^{-\int_s^{nt_0}K_1(r)\mathrm{d}r}\mathrm{d}s\\&\leq V(x_0)+\sum_{i=0}^{n-1}\int_{it_0}^{(i+1)t_0}|K_0(s)|\mathrm{e}^{-\int_{(i+1)t_0}^{nt_0}K_1(r)\mathrm{d}r}\mathrm{d}s\\&=V(x_0)+\bigg(\sum_{i=0}^{n-1}\mathrm{e}^{-(n-i-1)\lambda_0}\bigg)\int_0^{t_0}|K_0(s)|\mathrm{d}s,\quad n\geq 1,\end{aligned}$$

which is bounded in $n\geq 1$ since $\lambda_0:=\int_0^{t_0}K_1(t)\mathrm{d}t>0$. So, (4.11) holds. □

In the following example the SDE includes a class of fully non-dissipative models, for instance when $\nabla^{(1)}W\geq 0$, in the sense that

$$\sup_{|x-y|=r}\langle b_t(x,\mu)-b_t(y,\mu),x-y\rangle\geq 0,\quad r>0,\quad \mu\in\mathscr{P}.$$

Example 4.1. Let $\alpha\in C([0,t_0];(0,\infty))$, $b_0\in C^1(\mathbb{R}^d)$ with $b_0(x)=-|x|^{p-1}x$ for $|x|\geq 1$, and $W_t\in C^2(\mathbb{R}^d\times\mathbb{R}^d)$ measurable in $t\in[0,t_0]$ with

$$\begin{aligned}&\left\|\nabla^{(1)}\nabla^{(2)}W_t\right\|_\infty+\left\|\nabla^{(1)}W_t\right\|_\infty\leq\varepsilon\alpha_t,\\&\left\|\nabla^{(1)}\nabla^{(1)}W_t\right\|_\infty\leq\theta\alpha_t\end{aligned}\tag{4.12}$$

for some constant $\varepsilon>0$. We take t_0-periodic (b_t,σ_t) with

$$\begin{aligned}&b_t(x,\mu):=\alpha_t b_0(x)+\frac{\mu\big(\nabla^{(1)}W_t(x,\cdot)\big)}{1+\mu(V)},\\&\sigma_t:=\sqrt{\alpha_t}I_d,\quad (t,x,\mu)\in[0,t_0]\times\mathbb{R}^d\times\mathscr{P}_V,\end{aligned}$$

where $V(x):=\mathrm{e}^{|x|^p}$ for some $p\in[1/2,1]$. Moreover, let

$$\begin{aligned}&\tilde{\mathbb{W}}_V(\mu,\nu):=\inf_{\pi\in\mathscr{C}(\mu,\nu)}\int_{\mathbb{R}^d\times\mathbb{R}^d}(1\wedge|x-y|)\big(1+V(x)+V(y)\big)\pi(\mathrm{d}x,\mathrm{d}y),\\&\tilde{\mathbb{W}}_V(P_t^*\mu,\bar{\mu})\leq c\mathrm{e}^{-\delta t}\tilde{\mathbb{W}}_V(\mu,\bar{\mu}),\quad t\geq 0,\quad \mu,\nu\in\mathscr{P}_V.\end{aligned}\tag{4.13}$$

Then when $\varepsilon>0$ is small enough, there exist constants $c,\lambda>0$ such that

$$\tilde{\mathbb{W}}_V\big(P_{nt_0}^*\mu,\bar{\mu}\big)\leq c\mathrm{e}^{-\lambda n}\tilde{\mathbb{W}}_V(\mu,\bar{\mu}),\quad n\in\mathbb{N},\quad \mu\in\mathscr{P}_V.$$

Proof. It is easy to see that (H_7) with

$$K_0=\alpha_t\theta_0,\quad K_1(t)=\alpha_t\theta_1\tag{4.14}$$

holds for some constants $\theta_0,\theta_1>0$, (H_8) holds for $\hat{\sigma}=0$. Next, let $D_0:=\|\nabla b_0\|_\infty+\theta$ and let $l>0$ such that in (4.6)

$$k_{l,\beta}(t):=\inf_{|x-y|\geq l}\frac{\theta_1V(x)+\theta_1V(y)-2\theta_0}{\beta^{-1}+V(x)+V(y)}\geq k_0\alpha_t,\quad t\in[0,t_0]\tag{4.15}$$

holds for some constant $k_0>0$. Now, we take $\psi\in\Psi_l$ such that

$$2\psi''(r)+D_0\psi'(r)\leq -D_1\psi(r),\quad r\in[0,l]$$

holds for some constant $D_1>0$, for instance ψ and D_1 are the first mixed eigenfunction and eigenvalue of $2\mathrm{d}^2/\mathrm{d}r^2+D_0\mathrm{d}/\mathrm{d}r$ on $[0,l]$ with Dirichlet condition at 0 and Neumann condition at l.

Then the first inequality in (H_8) holds for

$$K_t:=\alpha_t D_0,\quad u_l(t):=D_1\alpha_t,\quad t\in[0,t_0]. \tag{4.16}$$

Moreover, noting that

$$|V(x)-V(y)|\le c_0\psi(|x-y|)\big(1+V(x)+V(y)\big)$$

holds for some constant $c_0>0$, we find a constant $c_1>0$ such that

$$\begin{aligned}&|b(x,\mu)-b(x,\nu)|\\ \le&\varepsilon\bigg(\frac{\big|\mu\big(\nabla^{(1)}W(x,\cdot)\big)-\nu\big(\nabla^{(1)}W(x,\cdot)\big)\big|}{1+\mu(V)\vee\nu(V)}+\frac{\|\nabla^{(1)}W\|_\infty|\mu(V)-\nu(V)|}{\big(1+\mu(V)\big)\big(1+\nu(V)\big)}\bigg)\\ \le&\frac{c_1\varepsilon\alpha_t\mathbb{W}_{\psi,V}(\mu,\nu)}{1+\mu(V)+\nu(V)}\le c_1\varepsilon\beta^{-1}\alpha_t\hat{\mathbb{W}}_{\psi,\beta V}(\mu,\nu).\end{aligned}$$

Combining this with $D_0:=\|\nabla b^0\|_\infty+\theta$ and (4.12), we obtain the second inequality in (H_8) for the above $K_t:=\alpha_t D_0$ and

$$\theta_t:=c_1\varepsilon\beta^{-1}\alpha_t,\quad t\in[0,t_0]. \tag{4.17}$$

Since (4.7) implies $\alpha_{l,\beta}(t)\to 0$ as $\beta\to 0$, by (4.8), (4.14)-(4.17), there exist constants $\beta,\varepsilon_0>0,k_1$ such that for any $\varepsilon\in(0,\varepsilon_0]$,

$$\lambda_{l,\beta}(t)-\theta_t\ge k_1,\quad t\in[0,t_0].$$

Then the desired assertion follows from Theorem 4.1 and the fact that

$$C^{-1}\tilde{\mathbb{W}}_V\le\mathbb{W}_{\psi,\beta V}\le C\mathbb{W}_{\mathrm{V}}$$

holds for some constant $C>1$. □

5 Extensions to reflecting McKean-Vlasov SDEs

In this section, we investigate the exponential ergodicity for the reflecting McKean-Vlasov SDE (1.2) on a convex domain D. By the convexity, the reflection on boundary does not make any trouble in the proofs of previous results on ergodicity, so that all these results work also for (1.2).

Let $T\partial D$ be the tangent space of ∂D, which is well defined when ∂D is C^1.

Theorem 5.1. *Let D be convex, $b,\sigma\in C([0,\infty)\times\bar{D}\times\mathscr{P}_2(\bar{D}))$, and in (H_1)-(H_3) we use $(\bar{D},\mathscr{P}_2(\bar{D}))$ to replace $(\mathbb{R}^d,\mathscr{P}_2)$, and in $(H_3)(2)$ assume further that ∂D is C^2 and there exists a measurable function $h:[0,\infty)\times\partial D\to[0,\infty)$ such that*

$$\langle\{\nabla_n(\sigma_t\sigma_t^*)\}v,v\rangle|_{\partial D}\ge 0,\quad (\sigma_t\sigma_t^*v-h_tv)|_{\partial D}=0,\quad v\in T\partial D,\quad t\ge 0. \tag{5.1}$$

Then assertions in Theorem 2.1 *holds for* (1.2) *replacing* (1.3).

Proof. By [13, Theorem 2.6], (H_1) implies that (1.2) is well-posed for distributions in $\mathscr{P}_2(\bar{D})$ and satisfies

$$\mathbb{W}_2(P_t^*\mu,P_t^*\nu)^2\le \mathrm{e}^{\int_0^t(K_1(s)+K_2(s))\mathrm{d}s}\mathbb{W}_2(\mu,\nu),\quad \mu,\nu\in\mathscr{P}_2(\bar{D}). \tag{5.2}$$

Let $x_0\in D$. Since D is convex, we have $\langle x-x_0,\mathbf{n}(x)\rangle\le 0$ for $x\in\partial D$, so that as in the proof of Theorem 2.1, by (H_1) and applying Itô's formula to $|X_t-x_0|^2$ for $X_0=x_0$, we obtain

$$\mathrm{d}|X_t-x_0|^2\le\left\{c+\left(K_1(t)+\frac{\lambda}{2t_0}\right)|X_t-x_0|^2+K_2(t)E|X_t-x_0|^2\right\}\mathrm{d}t+\mathrm{d}M_t$$

for some martingale M_t. Since $\lambda>0$, this and the proof leading to (2.25) gives the same estimate, so that by (5.2) we prove the first assertion.

Under (H_2) holds, by [13, Theorem 2.4], there exists a constant $c_1>0$ such that

$$\mathbb{W}_2\big(P_{t_0}^*\mu,P_{t_0}^*\nu\big)\le c_1\mathbb{H}(\mu|\nu),\quad \mu,\nu\in\mathscr{P}_2(\bar{D}). \tag{5.3}$$

So, as in the proof of Theorem 2.1, it remains to prove the Talagrand inequality

$$\mathbb{W}_2(\mu,\bar{\mu}_0)^2\le c_2\mathrm{Ent}(\mu|\bar{\mu}_0),\quad \mu\in\mathscr{P}_2(\bar{D}) \tag{5.4}$$

for some constant $c_2>0$.

When $(H_3)(1)$ holds, by the convexity of D, for $(\bar{X}_t^x,\bar{X}_t^y)$ solving the following SDE with $\bar{X}_0^x=x$, $\bar{X}_0^y=y$, $x,y\in\bar{D}$:

$$\mathrm{d}\bar{X}_t=b_t(\bar{X}_t,\bar{\mu}_0)\mathrm{d}t+\sigma_t\mathrm{d}W_t+\mathbf{n}(\bar{X}_t)\mathrm{d}l_t,$$

(H_1) implies

$$\mathrm{d}\big|\bar{X}_t^x-\bar{X}_t^y\big|^2\le K_1(t)\big|\bar{X}_t^x-\bar{X}_t^y\big|^2\mathrm{d}t,\quad t\ge 0,$$

so that

$$\big|\bar{X}_t^x-\bar{X}_t^y\big|^2\le \mathrm{e}^{\int_0^tK_1(s)\mathrm{d}s}|x-y|^2,\quad x,y\in\bar{D},\quad t\ge 0.$$

Thus, the associated $\bar{P}_t$ satisfies the gradient estimate (2.28). Then (5.4) holds as shown in the proof of Theorem 2.1.

When $(H_3)(2)$ holds, the corresponding proof in that of Theorem 2.1 also works provided Lemma 2.1(2) holds for (1.2). According to its proof it suffices to prove [5, Theorem 4.1] for (1.2), which is included in the following Lemma 5.1. The proof is complete. □

Let Γ_t^1 and Γ_t^2 be in (2.3) for σ_t, b_t not depending on μ on a convex C^2 domain $\bar{D}$ replacing $\mathbb{R}^d$, where $b_t(x)$ is C^1 in x, $\sigma_t(x)$ is C^1 in t and C^2 in x. Consider the reflecting SDE

$$\mathrm{d}X_{s,t} = b_t(X_{s,t})\mathrm{d}t + \sigma_t(X_{s,t})\mathrm{d}W_t + \mathbf{n}(X_t)\mathrm{d}l_t, \quad t \geq s. \tag{5.5}$$

Let $P_{s,t}^*\mu = \mathscr{L}_{X_{s,t}}$ for the solution with $\mathscr{L}_{X_{s,s}} = \mu$. The generator is

$$L_t := \frac{1}{2}\mathrm{tr}\{\sigma_t\sigma_t^*\nabla^2\} + b_t\cdot\nabla, \quad t \geq 0.$$

We have the following lemma, which extends [5, Theorem 4.1] to the reflecting case.

Lemma 5.1. *Let $\{\Gamma_t^i\}_{i=1,2,t\geq 0}$ be as in* (2.3) *on a convex C^2 domain $\bar{D}$ replacing $\mathbb{R}^d$ for σ_t, b_t not depending on μ, and let* (5.1) *hold. Let $\gamma \in L^1_{loc}([0,\infty);\mathbb{R})$ such that*

$$\Gamma_t^2(f,f) \geq \gamma_t \Gamma_t^1(f,f), \quad f \in C^3(\bar{D}). \tag{5.6}$$

Let $s \geq 0, q_s > 0$ and $\nu_s \in \mathscr{P}(\bar{D})$. If the log-Sobolev inequality

$$\nu_s(f^2\log f^2) \leq 4q_s\nu_s\big(\Gamma_1^s(f,f)\big), \quad f \in C_b^1(\bar{D}), \quad \nu_s(f^2) = 1 \tag{5.7}$$

holds, then for any $t > s$, $\nu_t := P_{s,t}^*\nu_s$ satisfies

$$\nu_t(f^2\log f^2) \leq 4q_t\nu_t\big(\Gamma_1^t(f,f)\big), \quad f \in C_b^1(\bar{D}), \quad \nu_t(f^2) = 1, \tag{5.8}$$

where

$$q_t := q_s \mathrm{e}^{-2\int_s^t \gamma_r \mathrm{d}r} + \int_s^t \mathrm{e}^{-2\int_r^t \gamma_u \mathrm{d}u}\mathrm{d}r, \quad t \geq s. \tag{5.9}$$

Proof. Let $P_{s,t}f(x) := (P_{s,t}^*\delta_x)(f)$. We first prove

$$\sqrt{\Gamma_{s_0}^1(P_{s_0,s_1}f, P_{s_0,s_1}f)} \leq \mathrm{e}^{-\int_{s_0}^{s_1}\gamma_t \mathrm{d}t} P_{s_0,s_1}\sqrt{\Gamma_{s_1}^1(f,f)}, \quad s_1 \geq s_0 \geq 0 \tag{5.10}$$

for $f \in C_b^\infty(\bar{D})$.

By the Bochner-Weitzenböck formula, the inequality (5.6) is equivalent to

$$\Gamma_t^2(f,f) \geq \gamma_t \Gamma_t^1(f,f) + \frac{\big|\nabla \Gamma_t^1(f,f)\big|^2}{4\Gamma_t^1(f,f)}, \quad f \in C^3(\mathbb{R}^d). \tag{5.11}$$

Next, since ∂D is C^2 and convex, the second fundamental form is non-negative, i.e.

$$\begin{aligned}&\mathbb{I}(\nabla f, \nabla f)(x) := -\langle \nabla_{\nabla f} \mathbf{n}, \nabla f\rangle(x) = \mathrm{Hess}_f(\mathbf{n}, \nabla f)(x) \geq 0,\\ &f \in C^2(\bar{D}), \quad Nf|_{\partial D} = 0, \quad x \in \partial D,\end{aligned}$$

where the second equality follows from $\nabla_{\nabla f}\langle \mathbf{n}, \nabla f\rangle|_{\partial D} = 0$ due to $\langle \mathbf{n}, \nabla f\rangle|_{\partial D} = 0$. Combining this with (5.1), we see that for any $f \in C_b^\infty(\bar{D}), s_1 \geq t \geq 0$, and $x \in \partial D$,

$$\begin{aligned}&\big\langle \mathbf{n}, \nabla \Gamma_s^1\big(P_{t,s_1} f^2, P_{t,s_1} f^2\big)\big\rangle(x)\\ &= \big\langle \{\nabla_{\mathbf{n}}(\sigma_t \sigma_t^*)\} \nabla P_{t,s_1} f^2, \nabla P_{t,s_1} f^2\big\rangle(x) + 2\mathrm{Hess}_{P_{t,s_1} f^2}\big(\mathbf{n}, (\sigma_t \sigma_t^*) \nabla P_{t,s_1} f^2\big)(x) \geq 0.\end{aligned}$$

Combining this with (5.11) and applying Itô's formula to (5.5), for any $s_1 > s_0 \geq 0$,

$$\begin{aligned}&\mathrm{d}\sqrt{\Gamma_t^1\big(P_{t,s_1} f^2, P_{t,s_1} f^2\big)}(X_{s_0,t})\\ &= \left\{\frac{(\partial_t \Gamma_t^1)/2\big(P_{t,s_1} f^2, P_{t,s_1} f^2\big) - \Gamma_t^1\big(L_t P_{t,s_1} f^2, P_{t,s_1} f^2\big)}{\sqrt{\Gamma_t^1\big(P_{t,s_1} f^2, P_{t,s_1} f^2\big)}}\right.\\ &\qquad \left. + L_t \sqrt{\Gamma_t^1\big(P_{t,s_1} f^2, P_{t,s_1} f^2\big)}\right\}(X_{s_0,t})\mathrm{d}t\\ &\qquad + \mathrm{d}M_t + \Big\langle \mathbf{n}, \nabla \sqrt{\Gamma_t^1\big(P_{t,s_1} f^2, P_{t,s_1} f^2\big)}(X_{s_0,t})\Big\rangle \mathrm{d}l_t\\ &\geq \left\{\frac{\Gamma_t^2\big(P_{t,s_1} f^2, P_{t,s_1} f^2\big)}{\Gamma_t^1\big(P_{t,s_1} f^2, P_{t,s_1} f^2\big)^{\frac{1}{2}}} - \frac{|\nabla \Gamma_t^1\big(P_{t,s_1} f^2, P_{t,s_1} f^2\big)|^2}{4\Gamma_t^1\big(P_{t,s_1} f^2, P_{t,s_1} f^2\big)^{\frac{3}{2}}}\right\}(X_{s_0,t})\mathrm{d}t + \mathrm{d}M_t\\ &\geq \gamma_t \sqrt{\Gamma_t^1\big(P_{t,s_1} f^2, P_{t,s_1} f^2\big)}(X_{s_0,t})\mathrm{d}t + \mathrm{d}M_t, \quad t \in [s_0, s_1]\end{aligned}$$

holds for some martingale $(M_t)_{t \in [s_0, s_1]}$. By Gronwall's lemma this implies (5.10).

By (5.10), the desired assertion follows from a standard semigroup argument we include below for completeness. Let $f \in C_b^2(\bar{D})$ with $\inf f^2 > 0$. By the chain rule and Schwarz inequality, (5.10) implies

$$\Gamma_s^1\Big(\sqrt{P_{s,t} f^2}, \sqrt{P_{s,t} f^2}\Big) = \frac{\Gamma_s^1\big(P_{s,t} f^2, P_{s,t} f^2\big)}{4P_{s,t} f^2}$$

$$\leq \frac{\mathrm{e}^{-2\int_s^t \gamma_r \mathrm{d}r}\Big(P_{s,t}\sqrt{\Gamma_t^1(f^2,f^2)}\Big)^2}{4P_{s,t}f^2} \leq \mathrm{e}^{-2\int_s^t \gamma_r \mathrm{d}r} P_{s,t}\Gamma_t^1(f,f), \quad t\geq s\geq 0. \tag{5.12}$$

So,

$$\begin{aligned}
&P_{s,t}(f^2\log f^2) - \big(P_{s,t}f^2\big)\log P_{s,t}f^2 \\
&= \int_s^t \frac{\mathrm{d}}{\mathrm{d}r} P_{s,r}\big\{\big(P_{r,t}f^2\big)\log\big(P_{r,t}f^2\big)\big\}\mathrm{d}r \\
&= \int_s^t P_{s,r}\frac{\Gamma_r^1\big(P_{r,t}f^2,P_{r,t}f^2\big)}{P_{r,t}f^2}\mathrm{d}r \leq 4\big(P_{s,t}\Gamma_t^1(f,f)\big)\int_s^t \mathrm{e}^{-2\int_r^t \gamma_u \mathrm{d}u}\mathrm{d}r.
\end{aligned}$$

Combining this with (5.7) and (5.12), we obtain

$$\begin{aligned}
\nu_t(f^2\log f^2) &= \nu_s\big(P_{s,t}(f^2\log f^2)\big) \\
&\leq 4\nu_s\big(P_{s,t}\Gamma_t^1(f,f)\big)\int_s^t \mathrm{e}^{-2\int_r^t \gamma_u \mathrm{d}u}\mathrm{d}r + \nu_s\big(\big(P_{s,t}f^2\big)\log\big(P_{s,t}f^2\big)\big) \\
&\leq 4\nu_t\big(\Gamma_t^1(f,f)\big)\int_s^t \mathrm{e}^{-2\int_r^t \gamma_u \mathrm{d}u}\mathrm{d}r + 4q_s\nu_s\Big(\Gamma_s^1\Big(\sqrt{P_{s,t}f^2},\sqrt{P_{s,t}f^2}\Big)\Big) \\
&\quad + \big(\nu_s\big(P_{s,t}f^2\big)\big)\log\big(\nu_s\big(P_{s,t}f^2\big)\big) \\
&\leq 4\nu_t\big(\Gamma_t^1(f,f)\big)\left(\int_s^t \mathrm{e}^{-2\int_r^t \gamma_u \mathrm{d}u}\mathrm{d}r + q_s\mathrm{e}^{-2\int_s^t \gamma_u \mathrm{d}u}\right) + \nu_t(f^2)\log\nu_t(f^2).
\end{aligned}$$

Therefore, (5.8) holds for q_t in (5.9). □

Finally, we have the following extensions of Theorems 3.1 and (4.1).

Theorem 5.2. *Let D be convex, use $\bar{D}$ replace $\mathbb{R}^d$ in (H_4)-(H_8), and in (H_7) we assume further $\langle \nabla V,\mathbf{n}\rangle|_{\partial D}\leq 0$. Then assertions in Theorems* 3.1 *and* 4.1 *hold for* (1.2) *replacing* (1.3).

Proof. The well-posedness of (1.2) as well as estimates (3.3) and (4.9) have been included in [13, Theorems 2.7, 2.8], so that the other assertions follow from the proofs of Theorems 3.1 and 4.1. Indeed, the proof of Theorem 3.1 has nothing to do with the reflection. Moreover, by Itô's formula, (4.3) and $\langle \mathbf{n},\nabla V\rangle|_{\partial D}\leq 0$, we derive

$$\mathrm{d}V(X_t) \leq \{K_0(t) - K_1(t)V(X_t)\}\mathrm{d}t + \mathrm{d}M_t$$

for some local martingale M_t, so that the proof of (4.11) works also for the present case. □

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