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L^2-Contraction and Asymptotic Stability of Large Shock for Scalar Viscous Conservation Laws

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Abstract. We investigate L^2-contraction and time-asymptotic stability of large shock for scalar viscous conservation laws with polynomial flux. For the flux $f(u)=u^p$ $(2\le p\le 4)$ in the regime of its strict convexity, we can prove L^2-contraction and time-asymptotic stability of arbitrarily large viscous shock profile in H^1-framework by using a-contraction method with time-dependent shift and suitable weight function, which answers a question in [Blochas and Cheng, arXiv2501.01537, 2025]. Additionally, if the initial perturbation belongs to L^1, then L^2 time-asymptotic decay rate $t^{-1/4}$ can be obtained.

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1 Introduction

We are concerned with L^2-contraction and time-asymptotic stability of arbitrarily

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large shock for the following scalar viscous conservation laws with polynomial flux:

$$\begin{cases} u_t+f(u)_x=u_{xx}, \quad f(u)=u^p, \quad (t,x)\in\mathbb{R}_+\times\mathbb{R}, \\ u(0,x)=u_0(x), \\ \lim\limits_{x\to\pm\infty} u_0(x)=u_\pm, \end{cases} \tag{1.1}$$

where $u_0(x)$ is the given initial data and $u_\pm\in\mathbb{R}$ are the prescribed far-field states. We focus on the case that the asymptotic state of the solution to (1.1) is the viscous shock wave. Therefore, it is assumed that $p>1$ such that $f(u)=u^p$ is strictly convex for $u>0$, and that

$$0<u_+<u_-. \tag{1.2}$$

Remark that for the special case $p=2$, that is, the classical Burgers equation, $f''(u)=2$ and $f(u)=u^2$ is always strictly convex for any $u\in\mathbb{R}$, and then we only need to assume that $u_+<u_-$.

It can be expected that the large-time asymptotic behavior of the solution to (1.1) and (1.2) is determined by the following viscous shock profile $U(x-st)$:

$$\begin{cases} -sU'+f(U)'=U'', \quad '=\dfrac{d}{d\xi}, \quad \xi=x-st, \\ U(\pm\infty)=u_\pm, \end{cases} \tag{1.3}$$

where s is the shock speed determined by the Rankine-Hugoniot condition

$$s=\frac{f(u_+)-f(u_-)}{u_+-u_-}. \tag{1.4}$$

Integrating (1.3) over $(\pm\infty,\xi]$, we can get the following first order ODE:

$$\begin{aligned} U'=h(U)&:=-s(U-u_\pm)+f(U)-f(u_\pm) \\ &=(U-u_\pm)\left[\frac{f(U)-f(u_\pm)}{U-u_\pm}-\frac{f(u_+)-f(u_-)}{u_+-u_-}\right]<0. \end{aligned} \tag{1.5}$$

Note that the strict convexity of the flux $f(u)$ implies the above decreasing monotonicity of the viscous shock profile $U(\xi)$. Moreover, the existence of the viscous shock profile $U(\xi)$ to (1.3) is standard and it is unique up to any constant translation.

The stability of viscous shock wave for conservation laws has been extensively studied since the pioneer works of Hopf [5], Il'in and Oleinik [8] for one-dimensional (1D) scalar equation. In 1976, Sattinger [25] introduced a semigroup