L^2-Contraction and Asymptotic Stability of Large Shock for Scalar Viscous Conservation Laws

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Abstract. We investigate L^2-contraction and time-asymptotic stability of large shock for scalar viscous conservation laws with polynomial flux. For the flux $f(u) = u^p$ $(2 \leq p \leq 4)$ in the regime of its strict convexity, we can prove L^2-contraction and time-asymptotic stability of arbitrarily large viscous shock profile in H^1-framework by using a-contraction method with time-dependent shift and suitable weight function, which answers a question in [Blochas and Cheng, arXiv2501.01537, 2025]. Additionally, if the initial perturbation belongs to L^1, then L^2 time-asymptotic decay rate $t^{-1/4}$ can be obtained.

1 Introduction

We are concerned with L^2-contraction and time-asymptotic stability of arbitrarily

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large shock for the following scalar viscous conservation laws with polynomial flux:

$$\begin{cases} u_t + f(u)_x = u_{xx}, \quad f(u) = u^p, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ u(0,x) = u_0(x), \\ \lim_{x \to \pm\infty} u_0(x) = u_\pm, \end{cases} \tag{1.1}$$

where $u_0(x)$ is the given initial data and $u_\pm \in \mathbb{R}$ are the prescribed far-field states. We focus on the case that the asymptotic state of the solution to (1.1) is the viscous shock wave. Therefore, it is assumed that $p > 1$ such that $f(u) = u^p$ is strictly convex for $u > 0$, and that

$$0 < u_+ < u_-. \tag{1.2}$$

Remark that for the special case $p = 2$, that is, the classical Burgers equation, $f''(u) = 2$ and $f(u) = u^2$ is always strictly convex for any $u \in \mathbb{R}$, and then we only need to assume that $u_+ < u_-$.

It can be expected that the large-time asymptotic behavior of the solution to (1.1) and (1.2) is determined by the following viscous shock profile $U(x - st)$:

$$\begin{cases} -sU' + f(U)' = U'', \quad ' = \dfrac{d}{d\xi}, \quad \xi = x - st, \\ U(\pm\infty) = u_\pm, \end{cases} \tag{1.3}$$

where s is the shock speed determined by the Rankine-Hugoniot condition

$$s = \frac{f(u_+) - f(u_-)}{u_+ - u_-}. \tag{1.4}$$

Integrating (1.3) over $(\pm\infty, \xi]$, we can get the following first order ODE:

$$\begin{aligned} U' = h(U) &:= -s(U - u_\pm) + f(U) - f(u_\pm) \\ &= (U - u_\pm)\left[\frac{f(U) - f(u_\pm)}{U - u_\pm} - \frac{f(u_+) - f(u_-)}{u_+ - u_-}\right] < 0. \end{aligned} \tag{1.5}$$

Note that the strict convexity of the flux $f(u)$ implies the above decreasing monotonicity of the viscous shock profile $U(\xi)$. Moreover, the existence of the viscous shock profile $U(\xi)$ to (1.3) is standard and it is unique up to any constant translation.

The stability of viscous shock wave for conservation laws has been extensively studied since the pioneer works of Hopf [5], Il'in and Oleinik [8] for one-dimensional (1D) scalar equation. In 1976, Sattinger [25] introduced a semigroup

approach to establish the stability of viscous shock waves to 1D parabolic equations, including (1.1), in certain weighted spaces. Then Matsumura and Nishihara [23] and Goodman [3] independently proved the time-asymptotic stability of viscous shock profile for 1D isentropic Navier-Stokes equations and viscous conservation laws with artificial viscosity respectively, under the zero mass conditions such that the anti-derivative method can be applied. Meanwhile, Nishihara [24] proved the point-wise stability of viscous shock to 1D Burgers equation by virtue of the Hopf-Cole transformation and Kawashima-Matsumura [17] further obtained the convergence rate for the viscous shock to scalar equation by weighted energy method and the stability of viscous shock to both 1D full Navier-Stokes-Fourier equations and the discrete Broadwell model system. Note that in all the above mentioned time-asymptotic results for viscous shock to the system case, the crucial zero mass conditions are imposed to the initial perturbations such that the anti-derivative variables can be well-defined. Then Liu [20], Szepessy and Xin [27], Liu and Zeng [21] removed the zero mass conditions in [3, 17, 23] by introducing the constant shift on the viscous shock and the coupled diffusion waves in the transverse characteristic fields. And Mascia and Zumbrun [22] established the spectral stability of viscous shocks for the 1D compressible Navier-Stokes system, relaxing the zero mass conditions to a slightly weaker spectral condition. More recently, Kang, Vasseur and Wang [15, 16] proved the generic Riemann solutions (containing viscous shock wave, rarefaction wave, and even viscous contact wave) to both barotropic compressible Navier-Stokes equations [15] and full compressible Navier-Stokes(-Fourier) equations [16] with the help of a-contraction method for the stability of viscous shock wave with time-dependent shift invented in [12]. Remark that both [15] and [16] solve the long-standing open problems for the time-asymptotic stability of the composite waves of Riemann profiles. In addition, Freistuhler and Serre [2] proved the L^1-stability of viscous shock waves (see also Serre [26]), and Kenig and Merle [18] proved L^p-stability ($1 \le p \le +\infty$) of viscous shock waves to scalar equation. For multi-dimensional case, Goodman [4] proved nonlinear stability of planar shock profile for viscous scalar conservation laws by using the anti-derivative techniques and the shift function depending on both the time and the transverse spatial variables. Then Humpherys, Lyng and Zumbrun [7] proved the spectral stability of the planar viscous Navier-Stokes shock by the numerical Evans-function method in $\mathbb{R}^3$ and one can refer to the survey paper by Zumbrun [29] for the related results and the references therein. Very recently, Wang and Wang [28] proved the time-asymptotic stability of planar weak shock profile to 3D barotropic compressible Navier-Stokes equations in $\mathbb{R} \times \mathbb{T}^2$ under general H^2-initial perturbations by using a-contraction method.

On the other hand, for scalar conservation laws with or without viscosity, Kružkov [19] proved the famous L^1-contraction stability in quite general case. However, Kružkov's theory is not valid for L^p-contraction (for any $p > 1$), in particular, in the physical L^2-norm. With the time-dependent shift $\mathbf{X}(t)$ and possible weight function a, Kang and Vasseur prove L^2-contraction of arbitrarily large shock to both 1D general inviscid system of hyperbolic conservation laws [11] and 1D viscous Burgers equation [12], and the latter result is extended by Kang [9] to scalar viscous conservation laws with general strictly convex flux, provided that the viscous shock wave strength was sufficiently small. Then Kang, Vasseur and Wang [14] prove L^2-contraction of large planar shock to scalar multi-dimensional viscous conservation laws by a special transformation and Kang and Vasseur [13] obtain L^2-contraction of small shock to 1D compressible barotropic Navier-Stokes equations. Kang-Oh [10] obtained L^2 decay for large perturbations of weak viscous shock for multi-dimensional Burgers equation. Note that a-contraction theory has been well established at the inviscid level uniformly in the shock amplitude [11], and at the viscous level only for small shocks [12] even with small perturbations, except Kang and Vasseur's work [12] on specific Burgers' equation for arbitrarily large shock. In fact, in the context of the viscous model, a-contraction results in the large shock setting (even in the scalar case) remain largely open. Very recently, Blochas and Cheng [1] showed that the a-contraction property fails for large shock to certain kinds of viscous conservation laws, which showcases a "viscous destabilization" effect in the sense that the a-contraction property is verified for the inviscid model for arbitrarily shock, but can fail for the viscous one, and raised the open question of whether this a-contraction property of large shock holds for scalar viscous conservation laws with polynomial fluxes.

In the present paper, we aim to address the question in [1] under small H^1 framework around the shock profile and further prove the time-asymptotic stability of this large shock with the time decay rate. Based on a-contraction method with suitably chosen time-dependent shift and the weight function a, crucially depending on the shock wave strength, we can establish L^2-contraction for any large shock to (1.1) with strictly convex fluxes of the polynomial form $f(u) = u^p$ ($2 \leq p \leq 4$) and suitably small H^1 perturbations, and further prove L^∞ time-asymptotic stability of this large shock profile with the time decay rate if the initial H^1-perturbation additionally lies in L^1. In fact, our proof is motivated by the recent work of Huang, Wang and Zhang [6] for the time-asymptotic stability of composite wave of large viscous Oleinik shock and rarefaction wave for the cubic non-convex scalar viscous conservation laws.

Precisely, our main result can be stated as follows.

Theorem 1.1. *For any given $u_\pm$ satisfying (1.2), let $U(\xi)$ be viscous shock wave defined in (1.3). Then there exists a positive constant ϵ^* such that if the initial data u_0 satisfies*

$$\|u_0(\cdot) - U(\cdot - x_0)\|_{H^1(\mathbb{R})} < \epsilon^* \tag{1.6}$$

for any initial shock location $x_0 \in \mathbb{R}$, then Cauchy problem (1.1) with $p \in [2,4]$ admits a unique global-in-time classical solution u. Moreover, there exists an absolutely continuous shift $\mathbf{X}(t)$ (defined in (2.5)) and a smooth weight function $a(U(\xi)) : \mathbb{R} \to \mathbb{R}_+$ (defined in (2.4)) such that the following L^2-contraction of arbitrarily large viscous shock holds

$$\frac{d}{dt} \int_{\mathbb{R}} a\big(U(x - st - \mathbf{X}(t))\big) \big|u(t,x) - U\big(x - st - \mathbf{X}(t)\big)\big|^2 dx \leq 0, \quad \forall t > 0. \tag{1.7}$$

Moreover, the time-asymptotic stability of large viscous shock holds

$$\lim_{t \to +\infty} \sup_{x \in \mathbb{R}} \big|u(t,x) - U\big(x - st - \mathbf{X}(t)\big)\big| = 0 \tag{1.8}$$

with

$$\lim_{t \to +\infty} |\dot{\mathbf{X}}(t)| = 0. \tag{1.9}$$

In addition, if $u_0(x) - U(x - x_0) \in L^1(\mathbb{R})$, we have the following L^2-time decay rate:

$$\big\|u(t,\cdot) - U\big(\cdot - st - \mathbf{X}(t)\big)\big\|_{L^2(\mathbb{R})} \leq \frac{C_* \|u_0(\cdot) - U(\cdot - x_0)\|_{L^2(\mathbb{R})}}{1 + C_* t^{1/4} \|u_0(\cdot) - U(\cdot - x_0)\|_{L^2(\mathbb{R})}}, \tag{1.10}$$

where

$$C_* := C\big[1 + \|u_0(\cdot) - U(\cdot - x_0)\|_{L^1(\mathbb{R})} + \|u_0(\cdot) - U(\cdot - x_0)\|_{H^1(\mathbb{R})}\big].$$

Remark 1.1. Since the weight function $a(U(\xi))$ is bounded from below and above, L^2-contraction (1.7) implies the uniform L^2-stability

$$\int_{\mathbb{R}} \big|u(t,x) - U\big(x - st - \mathbf{X}(t)\big)\big|^2 dx \leq C^* \int_{\mathbb{R}} |u_0(x) - U(x - x_0)|^2 dx, \quad \forall t > 0, \tag{1.11}$$

where the positive constant $C^* = C^*(u_+, u_-, p)$ is independent of the time $t > 0$.

Remark 1.2. The shift $\mathbf{X}(t)$ is proved to satisfy the time-asymptotic behavior (1.9), which implies

$$\lim_{t \to +\infty} \frac{\mathbf{X}(t)}{t} = 0,$$

that is, the shift $\mathbf{X}(t)$ grows at most sub-linearly with respect to the time t. There-

fore, the shifted viscous shock wave $U(x-st-\mathbf{X}(t))$ keeps the original traveling wave profile time-asymptotically.

Notations. Throughout this paper, several positive generic uniform-in-time constants are denoted by C. Denote $L^p(\mathbb{R})(1\leq p\leq +\infty)$ and $H^1(\mathbb{R})$ as the usual Lebesgue space and Sobolev space in $\mathbb{R}$ with the norm

$$\|f\|_{L^p(\mathbb{R})}:=\left(\int_{\mathbb{R}}|f|^p d\xi\right)^{\frac{1}{p}}, \quad 1\leq p<+\infty, \qquad \|f\|_{L^\infty(\mathbb{R})}:=\operatorname*{esssup}_{x\in\mathbb{R}}|f(x)|,$$

and

$$\|f\|:=\|f\|_{L^2(\mathbb{R})}, \quad \|f\|_{H^1(\mathbb{R})}:=\left(\|f\|^2+\|f_\xi\|^2\right)^{\frac{1}{2}}.$$

For any function $f:\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$ and the shift function $\mathbf{X}(t)$, we denote

$$f^{\pm\mathbf{X}}(t,\xi):=f\big(t,\xi\pm\mathbf{X}(t)\big).$$

2 Preliminaries and main result

In this section, we start with the construction of weight function $a(U(\xi))$ and shift function $\mathbf{X}(t)$. Then, we present the local existence of the solution and the uniform-in-time a priori estimates, whose proofs are given in the subsequent sections. Finally, we give the proof of Theorem 1.1 by the continuity arguments.

For convenience, we rewrite the Eq. (1.1) through the coordinates transformation $(t,x)\to(t,\xi=x-st)$, and then $u(t,\xi):=u(t,\xi+st)=u(t,x)$ satisfies

$$u_t-su_\xi+f(u)_\xi=u_{\xi\xi}. \tag{2.1}$$

And $U^{-\mathbf{X}}:=U(\xi-\mathbf{X}(t))$ satisfies

$$U_t^{-\mathbf{X}}+\dot{\mathbf{X}}(t)U_\xi^{-\mathbf{X}}-sU_\xi^{-\mathbf{X}}+f(U^{-\mathbf{X}})_\xi=U_{\xi\xi}^{-\mathbf{X}}. \tag{2.2}$$

Then the perturbation $\phi(t,\xi):=u(t,\xi)-U(\xi-\mathbf{X}(t))$ satisfies

$$\begin{cases} \phi_t-s\phi_\xi-\dot{\mathbf{X}}(t)U_\xi^{-\mathbf{X}}+[f(\phi+U^{-\mathbf{X}})-f(U^{-\mathbf{X}})]_\xi=\phi_{\xi\xi}, & \text{(2.3a)} \\ \phi(0,\xi)=\phi_0(\xi):=u_0(\xi)-U(\xi-x_0) & \text{(2.3b)} \end{cases}$$

for any initial shock location $x_0\in\mathbb{R}$. In the sequel, we always assume that $p\in[2,4]$ for the polynomial flux $f(u)=u^p$.

2.1 Construction of weight function

Define the weight function $a(U(\xi))$ as

$$a\big(U(\xi)\big) := \frac{(u_- - u_+)h(U)}{(U - u_+)(U - u_-)} = \frac{f(U) - f(u_-)}{U - u_-} - \frac{f(U) - f(u_+)}{U - u_+}. \tag{2.4}$$

It is easy to check that $a \in C^\infty(\mathbb{R}), a(U(\xi)) > c > 0$ for any $\xi \in \mathbb{R}$, that is, $U \in (u_+, u_-)$, and $\|a(U(\xi))\|_{C^1(\mathbb{R})} \leq C$. Notice that for the Burgers equation case ($p = 2$), the above weight function $a = u_- - u_+$ is exactly the shock wave strength, which means that there is no need for the weight and is consistent with the classical result [12] for Burgers equation.

2.2 Construction of shift function

Define the shift function $\mathbf{X}(t)$ as the solution to the following ODE:

$$\begin{cases} \dot{\mathbf{X}}(t) = -\dfrac{4}{(u_- - u_+)^2} \displaystyle\int_{\mathbb{R}} a\big(U^{-\mathbf{X}}(\xi)\big) U_\xi^{-\mathbf{X}}(\xi)\phi(t,\xi)d\xi, \\ \mathbf{X}(0) = x_0, \end{cases} \tag{2.5}$$

where the initial shock location x_0 can be chosen arbitrarily with or without zero mass condition.

Denote

$$\phi^{\mathbf{X}}(t,\xi) := \phi\big(t, \xi + \mathbf{X}(t)\big) = u\big(t, \xi + \mathbf{X}(t)\big) - U(\xi).$$

Then the perturbation equation (2.3) can be rewritten as

$$\phi_t^{\mathbf{X}} - \dot{\mathbf{X}}(t)\big(\phi_\xi^{\mathbf{X}} + U_\xi\big) - s\phi_\xi^{\mathbf{X}} + [f(\phi^{\mathbf{X}} + U) - f(U)]_\xi = \phi_{\xi\xi}^{\mathbf{X}}. \tag{2.6}$$

By (2.5), we have

$$\dot{\mathbf{X}}(t) = -\frac{4}{(u_- - u_+)^2} \int_{\mathbb{R}} a\big(U(\xi)\big) U_\xi(\xi)\phi^{\mathbf{X}}(t,\xi)d\xi. \tag{2.7}$$

Substituting (2.7) into (2.6) gives that

$$\phi_t^{\mathbf{X}} + \frac{4\big(\phi_\xi^{\mathbf{X}} + U_\xi\big)}{(u_- - u_+)^2} \int_{\mathbb{R}} a(U) U_\xi \phi^{\mathbf{X}} d\xi - s\phi_\xi^{\mathbf{X}} + [f(\phi^{\mathbf{X}} + U) - f(U)]_\xi = \phi_{\xi\xi}^{\mathbf{X}} \tag{2.8}$$

with the initial condition

$$\phi^{\mathbf{X}}(t = 0, \xi) = u_0(\xi + x_0) - U(\xi) := \phi_0^{\mathbf{X}}(\xi). \tag{2.9}$$

Note that both the non-local equation (2.8) for $\phi^{\mathbf{X}}$ and the initial value $\phi_0^{\mathbf{X}}(\xi)$

in (2.9) are independent of the definition of the shift function $\mathbf{X}(t)$, even though we still write as $\phi^{\mathbf{X}}$.

We reformulate the Cauchy problem at general initial time $\tau \geq 0$, that is

$$
\begin{cases}
\phi_t^{\mathbf{X}} + \dfrac{4(\phi_\xi^{\mathbf{X}} + U_\xi)}{(u_- - u_+)^2} \displaystyle\int_{\mathbb{R}} a(U) U_\xi \phi^{\mathbf{X}} d\xi - s\phi_\xi^{\mathbf{X}} \\
\qquad + [f(\phi^{\mathbf{X}} + U) - f(U)]_\xi = \phi_{\xi\xi}^{\mathbf{X}}, \quad t > \tau, \\
\phi^{\mathbf{X}}(t = \tau, \xi) := \phi_\tau^{\mathbf{X}}(\xi).
\end{cases}
\tag{2.10}
$$

The local existence of the strong solution

$$
\phi^{\mathbf{X}} \in C\big([\tau, \tau + t_0]; H^1(\mathbb{R})\big) \cap L^2\big(\tau, \tau + t_0; H^2(\mathbb{R})\big)
$$

to the scalar nonlocal equation (2.8) can be stated as follows, whose proof will be given in Appendix A.

Proposition 2.1. *(Local Existence) For any $M > 0$, there exists a positive constant $t_0 = t_0(M)$, which is independent of τ such that if $\|\phi_\tau^{\mathbf{X}}\|_{H^1(\mathbb{R})} \leq M$, then the Cauchy problem (2.8)-(2.9) has a unique strong solution $\phi^{\mathbf{X}}(t, \xi)$ on the time interval $[\tau, \tau + t_0]$ satisfying*

$$
\begin{cases}
\phi^{\mathbf{X}}(t, \xi) \in C\big([\tau, \tau + t_0]; H^1(\mathbb{R})\big) \cap L^2\big(\tau, \tau + t_0; H^2(\mathbb{R})\big), \\
\displaystyle\sup_{t \in [\tau, \tau + t_0]} \|\phi^{\mathbf{X}}(t, \cdot)\|_{H^1(\mathbb{R})} \leq 2M.
\end{cases}
\tag{2.11}
$$

Then we can determine a unique shift function $\mathbf{X}(t)$ by ODE (2.7) and the local solution $\phi(t, \xi) := u(t, \xi) - U(\xi - \mathbf{X}(t))$.

Proposition 2.2. *(A Priori Estimates) For any given $u_\pm$ satisfying (1.2), there exists a positive constant $\epsilon_0 > 0$ such that if the Cauchy problem (2.3) has a solution $\phi \in C([0, T]; H^1(\mathbb{R})) \cap L^2(0, T; H^2(\mathbb{R}))$ for some time $T > 0$ with*

$$
\mathcal{N}(T) := \sup_{0 \leq t \leq T} \|\phi\|_{H^1(\mathbb{R})} \leq \epsilon_0,
$$

then there exists a uniform-in-time positive constant C_0 such that for all $t \in [0, T]$,

$$
\|\phi(t)\|_{H^1(\mathbb{R})}^2 + \int_0^t \|\phi_\xi\|_{H^1(\mathbb{R})}^2 d\tau + \int_0^t \int_{\mathbb{R}} \phi^2 |U_\xi^{-\mathbf{X}}| d\xi d\tau
$$

$$
+ \int_0^t |\dot{\mathbf{X}}(\tau)|^2 d\tau \leq C_0 \|\phi_0\|_{H^1(\mathbb{R})}^2,
\tag{2.12}
$$

$$
|\dot{\mathbf{X}}(t)| \leq C_0 \|\phi\|_{L^\infty(\mathbb{R})}.
\tag{2.13}
$$

The proof of Proposition 2.2 will be given in next section. Based on Propositions 2.1 and 2.2, we can prove Theorem 1.1 as follows.

2.3   The continuity argument

By the continuity argument, we can extend the local solution to the global one for all $t \in [0,+\infty)$ as follows. Define

$$\epsilon^* := \min \left\{ \frac{\epsilon_0}{4}, \sqrt{\frac{\epsilon_0^2}{64C_0}} \right\}, \quad M := \frac{\epsilon_0}{4},$$

where ϵ_0 and C_0 are given in Proposition 2.2. By (1.6),

$$\left\| \phi_0^{\mathbf{X}} \right\|_{H^1(\mathbb{R})} = \| \phi_0 \|_{H^1(\mathbb{R})} < \epsilon^* \leq \frac{\epsilon_0}{4} \ (=M),$$

and using the local existence result in Proposition 2.1, there exists a positive constant $T_0 = T_0(M)$ such that a unique solution on $[0,T_0]$ satisfying

$$\left\| \phi^{\mathbf{X}}(t,\cdot) \right\|_{H^1(\mathbb{R})} \leq \frac{\epsilon_0}{2}, \quad t \in [0,T_0].$$

Note that
$$\| \phi(t,\cdot) \|_{H^1(\mathbb{R})} = \left\| \phi^{\mathbf{X}}(t,\cdot) \right\|_{H^1(\mathbb{R})} \leq \frac{\epsilon_0}{2}, \quad t \in [0,T_0].$$

Especially, since $\mathbf{X}(t)$ is absolutely continuous and $\phi^{\mathbf{X}} \in C([0,T_0];H^1(\mathbb{R}))$, we have $\phi \in C([0,T_0];H^1(\mathbb{R}))$. Consider the maximal existence time

$$T_M := \sup \left\{ t > 0 \mid \sup_{\tau \in [0,t]} \| \phi(\tau,\cdot) \|_{H^1(\mathbb{R})} \leq \epsilon_0 \right\}.$$

If $T_M < \infty$, then the continuity argument implies that

$$\sup_{\tau \in [0,T_M]} \| \phi(\tau,\cdot) \|_{H^1(\mathbb{R})} = \epsilon_0.$$

By Proposition 2.2, it holds that

$$\sup_{\tau \in [0,T_M]} \| \phi(\tau,\cdot) \|_{H^1(\mathbb{R})} \leq \sqrt{C_0 \| \phi_0 \|_{H^1(\mathbb{R})}^2} \leq \frac{\epsilon_0}{8},$$

which contradicts the fact $\sup_{\tau \in [0,T_M]} \| \phi(\tau,\cdot) \|_{H^1(\mathbb{R})} = \epsilon_0$. Therefore,

$$T_M = \infty.$$

By Proposition 3.1, we have

$$\frac{d}{dt}\int_{\mathbb{R}}a\big(U\big(x-st-\mathbf{X}(t)\big)\big)\big|u(t,x)-U\big(x-st-\mathbf{X}(t)\big)\big|^2 dx \leq 0, \quad \forall t>0, \qquad (2.14)$$

which verifies L^2-contraction in Theorem 1.1. By using Proposition 2.2 again, we have

$$\sup_{t>0}\|\phi(t,\cdot)\|_{H^1(\mathbb{R})}^2 + \int_0^\infty \|\phi_\xi\|_{H^1(\mathbb{R})}^2 d\tau$$

$$+ \int_0^\infty \int_{\mathbb{R}}\phi^2\big|U_\xi^{-\mathbf{X}}\big|d\xi d\tau \leq C_0\|\phi_0\|_{H^1(\mathbb{R})}^2, \qquad (2.15)$$

$$|\dot{\mathbf{X}}(t)| \leq C\|\phi(t,\cdot)\|_{L^\infty(\mathbb{R})}, \quad \forall t>0. \qquad (2.16)$$

2.4   Time-asymptotic behaviors

Now we justify the time-asymptotic behaviors (1.8) and (1.9). First we set

$$g(t) := \|\phi_\xi(t,\cdot)\|^2.$$

From (2.15), it is obvious that

$$g(t) \in L^1(0,+\infty).$$

Next we show that

$$g'(t) \in L^1(0,+\infty).$$

By (3.36)-(3.38), we have

$$\int_0^{+\infty}|g'(t)|dt = \int_0^{+\infty}\left|\frac{d}{dt}\|\phi_\xi\|^2\right|dt$$

$$\leq C\int_0^{+\infty}\left(\|\phi_\xi\|_{H^1(\mathbb{R})}^2 + |\dot{\mathbf{X}}(t)|^2 + \int_{\mathbb{R}}\phi^2\big|U_\xi^{-\mathbf{X}}\big|d\xi\right)dt < +\infty.$$

Hence,

$$\lim_{t\to+\infty}g(t) = \lim_{t\to+\infty}\|\phi_\xi(t,\cdot)\|^2 = 0. \qquad (2.17)$$

By Sobolev inequality, we have

$$\lim_{t\to+\infty}\|\phi(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq \lim_{t\to+\infty}\sqrt{2}\|\phi(t,\cdot)\|^{\frac{1}{2}}\|\phi_\xi(t,\cdot)\|^{\frac{1}{2}} = 0. \qquad (2.18)$$

By (2.16) and (2.18), it holds that

$$|\dot{\mathbf{X}}(t)| \leq C\|\phi(t,\cdot)\|_{L^\infty(\mathbb{R})} \to 0 \quad \text{as} \quad t \to +\infty. \qquad (2.19)$$

2.5 Decay estimate

Besides (1.6), if $u_0(x)-U(x-x_0)\in L^1(\mathbb{R})$, we first show that $\mathbf{X}(t)-x_0$ is uniformly bounded with respect to $t\in\mathbb{R}^+$. Rewrite $u(t,x)-U(x-st-\mathbf{X}(t))$ as follows:

$$u(t,x)-U\big(x-st-\mathbf{X}(t)\big)$$
$$=\underbrace{u(t,x)-U(x-st-x_0)}_{I_1(t,x)}+\underbrace{U(x-st-x_0)-U\big(x-st-\mathbf{X}(t)\big)}_{I_2(t,x)}. \qquad (2.20)$$

By L^1-contraction for solutions to the scalar viscous conservation laws [26], we have

$$\|I_1(t,\cdot)\|_{L^1(\mathbb{R})}\leq\|u_0(\cdot)-U(\cdot-x_0)\|_{L^1(\mathbb{R})}. \qquad (2.21)$$

Inspired by [12], we denote $\tau=\tau(t):=\mathbf{X}(t)-x_0$ and

$$\|I_2(t,\cdot)\|_{L^2(\mathbb{R})}^2=\int_{\mathbb{R}}|U(\xi-\tau)-U(\xi)|^2d\xi=:R(\tau).$$

Then we have

$$\frac{\partial R}{\partial\tau}(\tau)=-2\int_{\mathbb{R}}[U(\xi-\tau)-U(\xi)]\frac{\partial U(\xi-\tau)}{\partial\xi}d\xi$$

$$=-2\int_{\mathbb{R}}\int_{\xi}^{\xi-\tau}\frac{\partial U(y)}{\partial y}dy\frac{\partial U(\xi-\tau)}{\partial\xi}d\xi$$

$$=2\int_{\mathbb{R}}\int_{x}^{x+\tau}(U)'(y)(U)'(x)dydx,$$

which is strictly positive for $\tau>0$. Furthermore, if $\tau>1$, then we have

$$\frac{\partial R}{\partial\tau}(\tau)\geq2\int_{\mathbb{R}}\int_{x}^{x+1}(U)'(y)(U)'(x)dydx=:\beta>0.$$

Hence, for $\tau>1$,

$$R(\tau)\geq R(1)+\beta(\tau-1)\geq\beta(\tau-1),$$

and then it holds that

$$1<\tau\leq\frac{R(\tau)}{\beta}+1.$$

Since

$$R(\tau)=\int_{\mathbb{R}}|U(\xi-\tau)-U(\xi)|^2d\xi=\int_{\mathbb{R}}|U(\xi)-U(\xi+\tau)|^2d\xi=R(-\tau),$$

that is, $R(\tau)$ is an even function with respect to $\tau \in \mathbb{R}$, we have for any $\tau < -1$,

$$R(\tau) \geq R(1) + \beta(-\tau - 1) = R(1) + \beta(|\tau| - 1) \geq \beta(|\tau| - 1).$$

On the other hand, it is obvious that for $\tau \in [-1,1]$,

$$R(\tau) \geq 0 > \beta(|\tau| - 1).$$

Therefore, we have for any $\tau \in \mathbb{R}$,

$$|\tau| \leq \frac{R(\tau)}{\beta} + 1,$$

where

$$\beta := 2 \int_{\mathbb{R}} \int_x^{x+1} (U)'(y)(U)'(x) dy dx > 0.$$

Equivalently, it holds that for any $t \in \mathbb{R}_+$,

$$|\mathbf{X}(t) - x_0| \leq \frac{\|I_2(t,\cdot)\|_{L^2(\mathbb{R})}^2}{\beta} + 1. \tag{2.22}$$

Using the inequality $(a+b)^2 \geq a^2 - 2|ab|$ and (2.15), (2.21), we can get

$$\|I_2(t,\cdot)\|_{L^2(\mathbb{R})}^2 \leq 2 \int_{\mathbb{R}} |I_1(t,x)| |I_2(t,x)| dx + \int_{\mathbb{R}} (I_1(t,x) + I_2(t,x))^2 dx$$

$$\leq 2\|I_2(t,\cdot)\|_{L^\infty(\mathbb{R})} \|I_1(t,\cdot)\|_{L^1(\mathbb{R})} + \int_{\mathbb{R}} |u(t,x) - U(x - st - \mathbf{X}(t))|^2 dx$$

$$= 2\|I_2(t,\cdot)\|_{L^\infty(\mathbb{R})} \|I_1(t,\cdot)\|_{L^1(\mathbb{R})} + \int_{\mathbb{R}} \phi^2(t,\xi) d\xi$$

$$\leq 2|u_- - u_+| \|u_0(\cdot) - U(\cdot - x_0)\|_{L^1(\mathbb{R})} + C_0\|\phi_0\|_{H^1(\mathbb{R})}^2, \tag{2.23}$$

where in the last inequality we use the fact that $\|I_2(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq |u_- - u_+|$. By (2.22) and (2.23), we get that $\mathbf{X}(t) - x_0$ is uniformly bounded and satisfies

$$|\mathbf{X}(t) - x_0| \leq \frac{1}{\beta} \big(2|u_- - u_+| \|u_0(\cdot) - U(\cdot - x_0)\|_{L^1(\mathbb{R})} + C_0\|\phi_0\|_{H^1(\mathbb{R})}\big) + 1. \tag{2.24}$$

Since $U(x)$ is a monotone decreasing function, we have

$$\|I_2(t,\cdot)\|_{L^1(\mathbb{R})} = \text{sgn}(\tau(t)) \int_{\mathbb{R}} (U(\xi - \tau(t)) - U(\xi)) d\xi$$

$$= \text{sgn}(\tau(t)) \int_{\mathbb{R}} \int_0^{-\tau(t)} \partial_y U(\xi + y) dy d\xi$$

$$= - \int_{\mathbb{R}} \int_0^{|\tau(t)|} \partial_y U(\xi + y) dy d\xi.$$

Then, using Fubini's theorem to get

$$\|I_2(t,\cdot)\|_{L^1(\mathbb{R})} = |\mathbf{X}(t) - x_0||u_- - u_+|. \tag{2.25}$$

By (2.21), (2.24) and (2.25), we have

$$\begin{aligned}
\|\phi(t,\cdot)\|_{L^1(\mathbb{R})} &= \|(I_1 + I_2)(t,\cdot)\|_{L^1(\mathbb{R})} \\
&\leq \|I_1(t,\cdot)\|_{L^1(\mathbb{R})} + \|I_2(t,\cdot)\|_{L^1(\mathbb{R})} \leq C_1,
\end{aligned} \tag{2.26}$$

where

$$\begin{aligned}
C_1 &= \|u_0(\cdot) - U(\cdot - x_0)\|_{L^1(\mathbb{R})} \\
&\quad + |u_- - u_+|\left[\frac{1}{\beta}\left(2|u_- - u_+|\|u_0(\cdot) - U(\cdot - x_0)\|_{L^1(\mathbb{R})} + C_0\|\phi_0\|_{H^1(\mathbb{R})}\right) + 1\right].
\end{aligned}$$

Then Gagliardo-Nirenberg interpolation inequality shows that

$$\|\phi(t,\cdot)\|_{L^2(\mathbb{R})} \leq C\|\phi(t,\cdot)\|_{L^1(\mathbb{R})}^{\frac{2}{3}} \|\partial_x\phi(t,\cdot)\|_{L^2(\mathbb{R})}^{\frac{1}{3}}.$$

By (2.26), we have

$$\|\phi(t,\cdot)\|_{L^2(\mathbb{R})}^3 \leq C\|\phi(t,\cdot)\|_{L^1(\mathbb{R})}^2 \|\partial_x\phi(t,\cdot)\|_{L^2(\mathbb{R})} \leq C_2\|\partial_x\phi(t,\cdot)\|_{L^2(\mathbb{R})}. \tag{2.27}$$

From the proof of Propositions 3.1 and 3.2 (see (3.26)) and (3.34), for positive constant C we have

$$\frac{d}{dt}\int_{\mathbb{R}}(Ca+1)(\phi^{\mathbf{X}})^2 d\xi \leq -2\int_{\mathbb{R}}(\phi_\xi^{\mathbf{X}})^2 d\xi. \tag{2.28}$$

Since $0 < c < a^{-\mathbf{X}} < C$ for positive constants c and C, by (2.27) and (2.28), we have

$$\begin{aligned}
\frac{d}{dt}&\left\|\phi(t,\cdot)\sqrt{Ca^{-\mathbf{X}}+1}\right\|_{L^2(\mathbb{R})}^2 \\
&\leq -2\|\partial_x\phi(t,\cdot)\|_{L^2(\mathbb{R})}^2 \leq -\frac{2}{(C_2)^2}\|\phi(t,\cdot)\|_{L^2(\mathbb{R})}^6 \\
&\leq -C_3\left\|\phi(t,\cdot)\sqrt{Ca^{-\mathbf{X}}+1}\right\|_{L^2(\mathbb{R})}^6,
\end{aligned}$$

which implies the decay estimate

$$\left\|\phi(t,\cdot)\sqrt{Ca^{-\mathbf{X}}+1}\right\|_{L^2(\mathbb{R})}^4 \leq \frac{\left\|\phi_0\sqrt{Ca^{-\mathbf{X}}+1}\right\|_{L^2(\mathbb{R})}^4}{1+2C_3 t\left\|\phi_0\sqrt{Ca^{-\mathbf{X}}+1}\right\|_{L^2(\mathbb{R})}^4}. \tag{2.29}$$

By (2.29) and the inequality $2(a+b)^{1/4} \geq a^{1/4}+b^{1/4}$, we have

$$\|\phi(t,\cdot)\|_{L^2(\mathbb{R})} \leq \frac{C_* \|\phi_0\|_{L^2(\mathbb{R})}}{1+C_* t^{1/4} \|\phi_0\|_{L^2(\mathbb{R})}} \tag{2.30}$$

for some uniform-in-time positive constant C_*. Thus, we complete the proof of Theorem 1.1.

3 Uniform-in-time a priori estimates

In this section, we prove Proposition 2.2 for the uniform-in-time a priori estimates. To do this, we assume that the Cauchy problem (2.3) has a solution $\phi \in C([0,T];H^1(\mathbb{R})) \cap L^2(0,T;H^2(\mathbb{R}))$ for some constant $T>0$. First we prove L^2 relative entropy estimate for ϕ as follows.

Proposition 3.1. *There exists a positive constant $\epsilon_1 > 0$ such that if*

$$\mathcal{N}(T) := \sup_{0 \leq t \leq T} \|\phi(t,\cdot)\|_{H^1(\mathbb{R})} \leq \epsilon_1,$$

then for any $t \in [0,T]$, it holds that

$$\frac{d}{dt} \int_{\mathbb{R}} (\phi^{\mathbf{X}})^2 a d\xi \leq 0, \tag{3.1}$$

$$\|\phi(t,\cdot)\|_{L^2(\mathbb{R})}^2 + \int_0^t \int_{\mathbb{R}} |U_\xi^{-\mathbf{X}}| \phi^2 d\xi d\tau + \int_0^t |\dot{\mathbf{X}}(\tau)|^2 d\tau \leq C \|\phi_0\|_{L^2(\mathbb{R})}^2, \tag{3.2}$$

where the positive constant C is independent of T.

In order to prove Proposition 3.1, we need to use a Poincaré type inequality and weighted energy method. Thus, we start with a Poincaré type inequality.

Lemma 3.1 ([13]). *For any $f:[0,1] \to \mathbb{R}$ satisfying*

$$\int_0^1 y(1-y)|f'(y)|^2 dy < +\infty,$$

it holds that

$$\int_0^1 \left| f - \int_0^1 f dy \right|^2 dy \leq \frac{1}{2} \int_0^1 y(1-y)|f'(y)|^2 dy, \tag{3.3}$$

that is,

$$\int_0^1 f^2(y) dy - \left(\int_0^1 f(y) dy \right)^2 \leq \frac{1}{2} \int_0^1 y(1-y)|f'(y)|^2 dy. \tag{3.4}$$

For simplicity, we denote

$$a^{-\mathbf{X}} := a\big(U^{-\mathbf{X}}(\xi)\big) = a\big(U(\xi - \mathbf{X}(t))\big).$$

Multiplying the Eq. (2.3a) by $a^{-\mathbf{X}}\phi$, we can get

$$\left[\frac{1}{2}\phi^2 a^{-\mathbf{X}}\right]_t - s\phi_\xi \phi a^{-\mathbf{X}} + \left[f(\phi + U^{-\mathbf{X}}) - f(U^{-\mathbf{X}})\right]_\xi \phi a^{-\mathbf{X}}$$
$$+ \frac{1}{2}\dot{\mathbf{X}}(t)\phi^2 (a^{-\mathbf{X}})_\xi - \dot{\mathbf{X}}(t)U_\xi^{-\mathbf{X}}\phi a^{-\mathbf{X}} = \phi_{\xi\xi}\phi a^{-\mathbf{X}}. \tag{3.5}$$

Integrating (3.5) over $\mathbb{R}$ with respect to ξ and changing variable $\xi \to \xi - \mathbf{X}(t)$ and denoting $\phi^{\mathbf{X}} := \phi(t, \xi + \mathbf{X}(t))$, we have

$$\left[\frac{1}{2}\int_{\mathbb{R}}(\phi^{\mathbf{X}})^2 a\, d\xi\right]_t + \int_{\mathbb{R}}(\phi^{\mathbf{X}})^2 \left[\frac{s}{2}a_\xi - \frac{1}{2}a_{\xi\xi} - \frac{a_\xi}{2}f'(U) + \frac{af''(U)}{2}U_\xi\right]d\xi$$
$$+ \dot{\mathbf{X}}(t)\left[\frac{1}{2}\int_{\mathbb{R}}(\phi^{\mathbf{X}})^2 a_\xi\, d\xi - \int_{\mathbb{R}}\phi^{\mathbf{X}}aU_\xi\, d\xi\right]$$
$$+ \int_{\mathbb{R}}a(\phi_\xi^{\mathbf{X}})^2\, d\xi + \int_{\mathbb{R}}O(1)(\phi^{\mathbf{X}})^3 U_\xi\, d\xi = 0, \tag{3.6}$$

where we have used the fact

$$\int_{\mathbb{R}}a^{-\mathbf{X}}\phi[f(\phi + U^{-\mathbf{X}}) - f(U^{-\mathbf{X}})]_\xi\, d\xi$$
$$= \int_{\mathbb{R}}a\phi^{\mathbf{X}}[f(\phi^{\mathbf{X}} + U) - f(U)]_\xi\, d\xi$$
$$= \int_{\mathbb{R}}aU_\xi[f(\phi^{\mathbf{X}} + U) - f(U) - f'(U)\phi^{\mathbf{X}}]\, d\xi$$
$$+ \int_{\mathbb{R}}a_\xi\left[\int_U^{\phi^{\mathbf{X}}+U}f(\eta)\, d\eta - \phi^{\mathbf{X}}f(\phi^{\mathbf{X}}+U)\right]d\xi$$
$$= \int_{\mathbb{R}}(\phi^{\mathbf{X}})^2 U_\xi\left[\frac{af''(U)}{2} - \frac{a'f'(U)}{2}\right]d\xi + \int_{\mathbb{R}}O(1)(\phi^{\mathbf{X}})^3 U_\xi\, d\xi. \tag{3.7}$$

Let

$$y := \frac{U(\xi) - u_-}{u_+ - u_-},$$

then we have $\xi \in (-\infty, +\infty) \Longleftrightarrow y \in (0,1)$. Since $y_\xi = U_\xi/(u_+ - u_-) > 0$, there exists a unique inverse function $\xi = \xi(y)$ by the inverse function theorem. For any fixed $t > 0$, we denote

$$\psi(t,y) := \phi^{\mathbf{X}}\big(t, \xi(y)\big)a\big(U(\xi(y))\big). \tag{3.8}$$

In order to use the weighted Poincaré inequality with $\psi(t,y)$, we first have

$$-\frac{\dot{\mathbf{X}}(t)}{2}\int_{\mathbb{R}}\phi^{\mathbf{X}}aU_{\xi}d\xi=2\left(\int_{\mathbb{R}}\phi^{\mathbf{X}}a\frac{U_{\xi}}{u_{+}-u_{-}}d\xi\right)^{2}=2\left(\int_{0}^{1}\psi(t,y)dy\right)^{2}$$

$$\geq 2\int_{0}^{1}\psi^{2}(t,y)dy-\int_{0}^{1}|\psi_{y}(t,y)|^{2}y(1-y)dy. \tag{3.9}$$

Furthermore, we have

$$2\int_{0}^{1}\psi^{2}(t,y)dy=\frac{2}{u_{+}-u_{-}}\int_{\mathbb{R}}(\phi^{\mathbf{X}})^{2}a^{2}U_{\xi}d\xi, \tag{3.10}$$

and

$$\int_{0}^{1}|\psi_{y}(t,y)|^{2}y(1-y)dy$$

$$=\frac{1}{u_{+}-u_{-}}\int_{\mathbb{R}}\left[(\phi^{\mathbf{X}}a)_{\xi}\right]^{2}\frac{(U-u_{-})(u_{+}-U)}{U_{\xi}}d\xi=\int_{\mathbb{R}}\frac{\left[(\phi^{\mathbf{X}}a)_{\xi}\right]^{2}}{a}d\xi$$

$$=\int_{\mathbb{R}}a(\phi_{\xi}^{\mathbf{X}})^{2}d\xi-\int_{\mathbb{R}}\left(a_{\xi\xi}-\frac{a_{\xi}^{2}}{a}\right)(\phi^{\mathbf{X}})^{2}d\xi. \tag{3.11}$$

By (3.9)-(3.11), we have

$$-\frac{\dot{\mathbf{X}}(t)}{2}\int_{\mathbb{R}}\phi^{\mathbf{X}}aU_{\xi}d\xi+\int_{\mathbb{R}}a(\phi_{\xi}^{\mathbf{X}})^{2}d\xi\geq\int_{\mathbb{R}}\left[\frac{2a^{2}U_{\xi}}{u_{+}-u_{-}}+a_{\xi\xi}-\frac{a_{\xi}^{2}}{a}\right](\phi^{\mathbf{X}})^{2}d\xi. \tag{3.12}$$

By (2.4), we have $a_{\xi}=a'(U)U_{\xi}$ and $a_{\xi\xi}=a'(U)(f'(U)-s)U_{\xi}+a''(U)U_{\xi}^{2}$. Then from (3.6) and (3.12), we can get

$$\left[\frac{1}{2}\int_{\mathbb{R}}(\phi^{\mathbf{X}})^{2}ad\xi\right]_{t}+\int_{\mathbb{R}}(\phi^{\mathbf{X}})^{2}U_{\xi}\left[\frac{2a^{2}}{u_{+}-u_{-}}+\frac{af''(U)}{2}+\left(\frac{a''}{2}-\frac{(a')^{2}}{a}\right)h(U)\right]d\xi$$

$$+\frac{1}{2}\dot{\mathbf{X}}(t)\int_{\mathbb{R}}(\phi^{\mathbf{X}})^{2}a_{\xi}d\xi+\frac{(u_{-}-u_{+})^{2}}{8}|\dot{\mathbf{X}}(t)|^{2}+\int_{\mathbb{R}}O(1)|(\phi^{\mathbf{X}})^{3}U_{\xi}|d\xi\leq 0, \tag{3.13}$$

where we have used the fact

$$-\frac{\dot{\mathbf{X}}(t)}{2}\int_{\mathbb{R}}\phi^{\mathbf{X}}aU_{\xi}d\xi=\frac{(u_{-}-u_{+})^{2}}{8}|\dot{\mathbf{X}}(t)|^{2}.$$

Note that

$$g(U)=\frac{2a^{2}}{u_{+}-u_{-}}+\frac{af''(U)}{2}+\left[\frac{a''}{2}-\frac{(a')^{2}}{a}\right]h(U). \tag{3.14}$$

By direct calculations, we can obtain

$$
\begin{aligned}
g(U) &= -\frac{|u_+ - u_-|}{(U-u_-)^2(u_+-U)^2}\big[(U-u_-)(u_+-U)\big(hh''-(h')^2\big)\\
&\qquad\qquad\qquad\qquad + 3h^2 + hh'(u_+ + u_- - 2U)\big]\\
&= -\frac{|u_+ - u_-|}{(U-u_-)^2(u_+-U)^2}m(U),
\end{aligned} \tag{3.15}
$$

where

$$
m(U) := (U-u_-)(u_+-U)\big(hh''-(h')^2\big)+3h^2+hh'(u_++u_--2U), \tag{3.16}
$$

and $h = h(U)$ is defined in (1.5).

Lemma 3.2. *There exist positive constants $\beta = \beta(u_+,u_-,p)$ such that for any $U \in [u_+,u_-]$, it holds that $g(U) \le -\beta < 0$.*

Proof. By (2.4), we have

$$
a(u_+) = s - f'(u_+) = \frac{f(u_+)-f(u_-)}{u_+-u_-} - f'(u_+) > 0.
$$

By (3.14), we have

$$
\begin{aligned}
g(u_+) &= 2a(u_+)\left[\frac{f''(u_+)}{4} + \frac{a(u_+)}{u_+-u_-}\right]\\
&= \frac{2a(u_+)}{u_- - u_+}\left[\frac{f''(u_+)(u_- - u_+)}{4} + f'(u_+) - \frac{f(u_+)-f(u_-)}{u_+-u_-}\right]\\
&= -\frac{2a(u_+)}{(u_--u_+)^2}\left[f(u_-)-f(u_+)-f'(u_+)(u_--u_+)-\frac{f''(u_+)}{4}(u_--u_+)^2\right]\\
&= -\frac{2a(u_+)}{(u_--u_+)^2}\left[\frac{f''(u_+)}{4}(u_--u_+)^2+\frac{f'''(\theta)}{6}(u_--u_+)^3\right] < 0, \tag{3.17}
\end{aligned}
$$

where $\theta \in (u_+,u_-)$. On the other hand, by (2.4),

$$
a(u_-) = f'(u_-) - s = f'(u_-) - \frac{f(u_+)-f(u_-)}{u_+-u_-} > 0,
$$

then we have

$$
\begin{aligned}
g(u_-) &= 2a(u_-)\left[\frac{f''(u_-)}{4} + \frac{a(u_-)}{u_+-u_-}\right]\\
&= 2a(u_-)\left[u_-^{p-2}\left(\frac{p^2-5p+4}{4}+\frac{(2-p)u_+}{u_--u_+}+\frac{u_+^2}{(u_--u_+)^2}\right)-\frac{u_+^p}{(u_--u_+)^2}\right]
\end{aligned}
$$

$$\leq 2a(u_-)\left[u_-^{p-2}\left(\frac{(2-p)u_+}{u_--u_+}+\frac{u_+^2}{(u_--u_+)^2}\right)-\frac{u_+^p}{(u_--u_+)^2}\right]$$

$$=\frac{2a(u_-)}{(u_--u_+)^2}\underbrace{\left[(2-p)u_+(u_--u_+)u_-^{p-2}+u_+^2\left(u_-^{p-2}-u_+^{p-2}\right)\right]}_{=:l_1(u_-,u_+)}<0, \qquad (3.18)$$

where we have used the fact that for any $p\in[2,4]$, for any $0<u_+<u_-$,

$$l_1(u_-,u_+)<0.$$

Next, we claim that

$$\forall U\in(u_+,u_-),\quad g(U)<0. \qquad (3.19)$$

To prove the claim (3.19), by (3.16), we first have

$$m(U)=\underbrace{[h+(u_--U)h'][h+(u_+-U)h']}_{I_1}+\underbrace{(U-u_-)(u_+-U)hh''+2h^2}_{I_2}. \qquad (3.20)$$

First we estimate I_1. Since for any $U\in(u_+,u_-)$, we have

$$\begin{aligned}
&h(U)+(u_--U)h'(U)\\
&=-[h(u_-)-h(U)-(u_--U)h'(U)]\\
&=-\frac{h''(U)}{2}(u_--U)^2-\frac{h'''(\bar U)}{6}(u_--U)^3\\
&\leq-\frac{h''(U)}{2}(u_--U)^2<0,
\end{aligned} \qquad (3.21)$$

where $\bar U\in[U,u_-]$ and then $h'''(\bar U)=f'''(\bar U)>0$ for any $p\in[2,4]$.

On the other hand, we can calculate that

$$h(U)+(u_+-U)h'(U)+\frac{h''(U)}{4}(u_+-U)^2 \qquad (3.22)$$

$$=f'(U)(u_+-U)+f(U)-f(u_+)+\frac{f''(U)}{4}(u_+-U)^2$$

$$=u_+^p\left[\frac{(p-1)(p-4)}{4}\left(\frac{U}{u_+}\right)^p+\frac{3p-p^2}{2}\left(\frac{U}{u_+}\right)^{p-1}+\frac{p(p-1)}{4}\left(\frac{U}{u_+}\right)^{p-2}-1\right].$$

Set $z:=U/u_+$. Then we have $U\in(u_+,u_-)\Longleftrightarrow z\in(1,u_-/u_+)$. Define

$$H(z):=\frac{(p-1)(p-4)}{4}z^p+\frac{3p-p^2}{2}z^{p-1}+\frac{p(p-1)}{4}z^{p-2}-1.$$

It is obvious to compute that

$$H'(z) = \frac{p(p-1)}{4} z^{p-3} [(p-4)z^2 + 2(3-p)z + (p-2)].$$

Since for any $z \in (1, u_-/u_+)$, it holds that $(p-4)z^2 + 2(3-p)z + (p-2) < 0$. Hence, for any $p \in [2,4]$, we can get $H'(z) < 0$ for any $z \in (1, u_-/u_+)$. Therefore, we have $H(z) < H(1) = 0$ for any $z \in (1, u_-/u_+)$, which and (3.22) imply that for any $p \in [2,4]$,

$$h(U) + (u_+ - U)h'(U) < -\frac{h''(U)}{4}(u_+ - U)^2 < 0, \quad \forall U \in (u_+, u_-). \tag{3.23}$$

Therefore, by (3.21) and (3.23), we have for any $p \in [2,4]$, for any $U \in (u_+, u_-)$,

$$I_1 = [h + (u_- - U)h'][h + (u_+ - U)h'] > \frac{(h'')^2}{8}(u_+ - U)^2(u_- - U)^2. \tag{3.24}$$

By (3.20) and (3.24), we have for any $p \in [2,4]$,

$$m(U) = I_1 + I_2 > \frac{(h'')^2}{8}(u_+ - U)^2(u_- - U)^2 + (U - u_-)(u_+ - U)hh'' + 2h^2,$$

$$= 2\left[h + \frac{h''(U)(U-u_-)(u_+ - U)}{4}\right]^2 \geq 0, \quad \forall U \in (u_+, u_-). \tag{3.25}$$

Hence, by (3.15), we proved the claim (3.19). From (3.17) and (3.18), we have $g(u_\pm) < 0$. By the properties of continuous function over closed interval, there exists a positive constant $\beta = \beta(u_-, u_+, p)$ such that

$$g(U) \leq -\beta, \quad \forall U \in [u_+, u_-].$$

The proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Thus, by (3.13), we have

$$\left[\frac{1}{2}\int_{\mathbb{R}} (\phi^X)^2 a d\xi\right]_t + \beta \int_{\mathbb{R}} (\phi^X)^2 |U_\xi| d\xi + \frac{1}{2}\dot{X}(t)\int_{\mathbb{R}} (\phi^X)^2 a_\xi d\xi$$

$$+ \frac{(u_- - u_+)^2}{8}|\dot{X}(t)|^2 + \int_{\mathbb{R}} O(1)(\phi^X)^3 U_\xi d\xi \leq 0. \tag{3.26}$$

On the other hand, we have

$$\left|\frac{1}{2}\dot{X}(t)\int_{\mathbb{R}} (\phi^X)^2 a_\xi d\xi\right|$$

$$\leq \frac{4}{(u_- - u_+)^2} \|\phi^{\mathbf{X}}\|_{L^\infty(\mathbb{R})} \|a\|_{L^\infty(\mathbb{R})} \|a'\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |U_\xi| d\xi \int_{\mathbb{R}} (\phi^{\mathbf{X}})^2 |U_\xi| d\xi$$

$$\leq C\|\phi\|_{H^1(\mathbb{R})} \int_{\mathbb{R}} (\phi^{\mathbf{X}})^2 |U_\xi| d\xi \leq C\epsilon_1 \int_{\mathbb{R}} (\phi^{\mathbf{X}})^2 |U_\xi| d\xi. \tag{3.27}$$

By (3.26) and (3.27), we have

$$\left[\frac{1}{2}\int_{\mathbb{R}} (\phi^{\mathbf{X}})^2 a \, d\xi\right]_t + (\beta - C\epsilon_1)\int_{\mathbb{R}} (\phi^{\mathbf{X}})^2 |U_\xi| d\xi + \frac{(u_- - u_+)^2}{8} |\dot{\mathbf{X}}(t)|^2 \leq 0. \tag{3.28}$$

Integrating (3.28) with respect to t, then changing of variable $\xi \to \xi - \mathbf{X}(t)$, and choosing the suitable smallness of ϵ_1, we can prove Proposition 3.1.

Proposition 3.2. *There exists a positive constant $\epsilon_2 > 0$ such that if*

$$\mathcal{N}(T) := \sup_{0 \leq t \leq T} \|\phi(t, \cdot)\|_{H^1(\mathbb{R})} \leq \epsilon_2,$$

then for any $t \in [0, T]$, it holds that

$$\|\phi(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \int_0^t \|\phi_\xi(\tau, \cdot)\|_{L^2(\mathbb{R})}^2 d\tau$$

$$+ \int_0^t \int_{\mathbb{R}} |U_\xi^{-\mathbf{X}}| \phi^2 d\xi d\tau + \int_0^t |\dot{\mathbf{X}}(\tau)|^2 d\tau \leq C\|\phi_0\|_{L^2(\mathbb{R})}^2, \tag{3.29}$$

where the positive constant C is independent of T.

Proof. Multiplying the Eq. (2.3a) by ϕ, we can get

$$\left(\frac{\phi^2}{2}\right)_t - s\phi_\xi \phi + [f(\phi + U^{-\mathbf{X}}) - f(U^{-\mathbf{X}})]_\xi \phi - \dot{\mathbf{X}}(t) U_\xi^{-\mathbf{X}} \phi = \phi_{\xi\xi} \phi. \tag{3.30}$$

Integrating (3.30) over $\mathbb{R}$ with respect to ξ and changing variable $\xi \to \xi - \mathbf{X}(t)$ and denoting $\phi^{\mathbf{X}} := \phi(t, \xi + \mathbf{X}(t))$, we have

$$\left[\frac{1}{2}\int_{\mathbb{R}} (\phi^{\mathbf{X}})^2 d\xi\right]_t + \int_{\mathbb{R}} (\phi^{\mathbf{X}})^2 \frac{f''(U)}{2} U_\xi d\xi - \dot{\mathbf{X}}(t) \int_{\mathbb{R}} \phi^{\mathbf{X}} U_\xi d\xi$$

$$+ \int_{\mathbb{R}} (\phi_\xi^{\mathbf{X}})^2 d\xi + \int_{\mathbb{R}} O(1)(\phi^{\mathbf{X}})^3 U_\xi d\xi = 0, \tag{3.31}$$

where we have used the fact

$$\int_{\mathbb{R}} \phi[f(\phi + U^{-\mathbf{X}}) - f(U^{-\mathbf{X}})]_\xi d\xi$$

$$= \int_{\mathbb{R}} [f(\phi^{\mathbf{X}}+U)-f(U)]_\xi d\xi$$

$$= \int_{\mathbb{R}} U_\xi [f(\phi^{\mathbf{X}}+U)-f(U)-f'(U)\phi^{\mathbf{X}}]d\xi$$

$$= \int_{\mathbb{R}} (\phi^{\mathbf{X}})^2 U_\xi \frac{f''(U)}{2} + \int_{\mathbb{R}} O(1)(\phi^{\mathbf{X}})^3 U_\xi d\xi. \tag{3.32}$$

By Cauchy inequality and Hölder inequality, we have

$$-\dot{\mathbf{X}}(t)\int_{\mathbb{R}} \phi^{\mathbf{X}} U_\xi d\xi \leq \frac{|\dot{\mathbf{X}}(t)|^2}{2} + \frac{\left(\int_{\mathbb{R}} \phi^{\mathbf{X}} U_\xi d\xi\right)^2}{2}$$

$$\leq \frac{|\dot{\mathbf{X}}(t)|^2}{2} + \frac{|u_- - u_+|}{2}\int_{\mathbb{R}} (\phi^{\mathbf{X}})^2 U_\xi d\xi. \tag{3.33}$$

Hence, it holds that

$$\left[\frac{1}{2}\int_{\mathbb{R}} (\phi^{\mathbf{X}})^2 d\xi\right]_t + \int_{\mathbb{R}} (\phi^{\mathbf{X}}_\xi)^2 d\xi \leq C\left[\int_{\mathbb{R}} (\phi^{\mathbf{X}})^2|U_\xi|d\xi + |\dot{\mathbf{X}}(t)|^2\right]. \tag{3.34}$$

Integrating (3.34) with respect to ξ, and combining with Proposition 3.1 we complete the proof. $\quad\square$

Proposition 3.3. *There exists a positive constant $\epsilon_3 > 0$ such that if*

$$\mathcal{N}(T) := \sup_{0 \leq t \leq T} \|\phi(t,\cdot)\|_{H^1(\mathbb{R})} \leq \epsilon_3,$$

then there exists a uniform-in-time positive constant C such that for all $t \in [0,T]$, it holds that

$$\|\phi_\xi(t,\cdot)\|^2 + \int_0^t \|\phi_{\xi\xi}(\tau,\cdot)\|^2 d\tau \leq C\|\phi_0\|^2_{H^1(\mathbb{R})}. \tag{3.35}$$

Proof. Multiplying (2.3) by $-\phi_{\xi\xi}$ and integrating the resulted equation with respect to ξ, we can obtain

$$\frac{1}{2}\frac{d}{dt}\|\phi_\xi\|^2 + \|\phi_{\xi\xi}\|^2 - \dot{\mathbf{X}}(t)\int_{\mathbb{R}} \phi_{\xi\xi} U^{-\mathbf{X}}_\xi d\xi$$

$$= \int_{\mathbb{R}} (f(\phi+U^{-\mathbf{X}})-f(U^{-\mathbf{X}}))_\xi \phi_{\xi\xi} d\xi. \tag{3.36}$$

On one hand, we have

$$\int_{\mathbb{R}} |(f(\phi+U^{-\mathbf{X}})-f(U^{-\mathbf{X}}))_\xi|^2 d\xi$$

$$\leq 2\int_{\mathbb{R}}\left|\left(f'(\phi+U^{-\mathbf{X}})-f'(U^{-\mathbf{X}})\right)U_{\xi}^{-\mathbf{X}}\right|^2 d\xi + 2\int_{\mathbb{R}}|f'(\phi+U^{-\mathbf{X}})\phi_{\xi}|^2 d\xi$$

$$\leq C\int_{\mathbb{R}}\phi^2(U_{\xi}^{-\mathbf{X}})^2 d\xi + C\|\phi_{\xi}\|^2 \leq C\int_{\mathbb{R}}\phi^2|U_{\xi}^{-\mathbf{X}}|d\xi + C\|\phi_{\xi}\|^2. \tag{3.37}$$

On the other hand, we have

$$\left|\dot{\mathbf{X}}(t)\int_{\mathbb{R}}\phi_{\xi\xi}U_{\xi}^{-\mathbf{X}}d\xi\right| \leq \frac{1}{2}\|\phi_{\xi\xi}\|^2 + C|\dot{\mathbf{X}}(t)|^2. \tag{3.38}$$

By Proposition 3.2 and (3.36)-(3.38), we prove Proposition 3.3. $\qquad\square$

Therefore, Propositions 3.2 and 3.3 complete the proof of Proposition 2.2.

Appendix A

Proof of Proposition 2.1. Since both the non-local equation (2.8) for $\phi^{\mathbf{X}}$ and the initial value $\phi_0^{\mathbf{X}}(\xi)$ in (2.9) are independent of the definition of the shift function $\mathbf{X}(t)$, we will omit the superscript $\mathbf{X}$ for simplicity in this subsection. For any initial time $\tau \geq 0$, rewrite the Cauchy problem (2.10) as

$$\begin{cases} \phi_t - s\phi_{\xi} + \dfrac{4(\phi_{\xi}+U_{\xi})}{(u_--u_+)^2}\displaystyle\int_{\mathbb{R}}a(U)U_{\xi}\phi d\xi + [f(\phi+U)-f(U)]_{\xi} = \phi_{\xi\xi}, & t > \tau, \\ \phi(t=\tau,\xi) = \phi_{\tau}(\xi). \end{cases} \tag{A.1}$$

For any $M > 0$ with $\|\phi_{\tau}\|_{H^1(\mathbb{R})} \leq M$, define the solution space on the time interval $[\tau, \tau+t_0]$ for $t_0 \geq 0$ as

$$Y_{\tau,t_0}^M := \left\{\phi \in C\big([\tau,\tau+t_0];H^1(\mathbb{R})\big) \cap L^2\big(\tau,\tau+t_0;H^2(\mathbb{R})\big) \,\Big|\, \sup_{\tau \leq t \leq \tau+t_0}\|\phi\|_{H^1(\mathbb{R})} \leq M\right\}.$$

Now we define a mapping $\mathcal{T}$ on the solution space Y_{τ,t_0}^{2M}. For any $\phi(t,\xi) \in Y_{\tau,t_0}^{2M}$, define $\tilde{\phi}(t,\xi) := \mathcal{T}\phi(t,\xi)$ be the solution to the linear equation

$$\begin{cases} \tilde{\phi}_t - s\tilde{\phi}_{\xi} - \tilde{\phi}_{\xi\xi} = -\dfrac{4(\phi_{\xi}+U_{\xi})}{(u_--u_+)^2}\displaystyle\int_{\mathbb{R}}a(U)U_{\xi}\phi d\xi - [f(\phi+U)-f(U)]_{\xi}, & t > \tau, \quad \text{(A.2a)} \\ \tilde{\phi}(t=\tau,\xi) = \phi_{\tau}(\xi). & \text{(A.2b)} \end{cases}$$

First we prove that $\tilde{\phi}(t,\xi) := \mathcal{T}\phi(t,\xi) \in Y_{\tau,t_0}^{2M}$ for suitably small $t_0 = t_0(M) > 0$. Multiplying the Eq. (A.2a) by $\tilde{\phi}(t,\xi)$ and integrating the resulting equation with respect

to t and ξ over $[\tau, \tau + t_0] \times \mathbb{R}$, we can arrive at

$$\frac{1}{2}\|\tilde{\phi}\|^2(\tau + t_0) + \int_\tau^{\tau + t_0} \|\tilde{\phi}_\xi\|^2 dt$$

$$= \frac{1}{2}\|\phi_\tau\|^2 - \int_\tau^{\tau + t_0} \int_{\mathbb{R}} \tilde{\phi}[f(\phi + U) - f(U)]_\xi d\xi dt$$

$$- 4 \int_\tau^{\tau + t_0} \int_{\mathbb{R}} \frac{\tilde{\phi}(\phi_\xi + U_\xi)}{(u_- - u_+)^2} d\xi \int_{\mathbb{R}} a(U)U_\xi \phi d\xi dt. \tag{A.3}$$

Now we estimate the last two terms on the right-hand side of (A.3). First it holds that

$$\left| - \int_\tau^{\tau + t_0} \int_{\mathbb{R}} \tilde{\phi}[f(\phi + U) - f(U)]_\xi d\xi dt \right|$$

$$= \left| \int_\tau^{\tau + t_0} \int_{\mathbb{R}} \tilde{\phi}[f'(\phi + U)\phi_\xi + (f'(\phi + U) - f'(U))U_\xi] d\xi dt \right|$$

$$= \left| \int_\tau^{\tau + t_0} \int_{\mathbb{R}} \tilde{\phi}[f'(\phi + U)\phi_\xi + f''(\theta_1)U_\xi\phi] d\xi dt \right|$$

$$\leq C(M) \int_\tau^{\tau + t_0} \|\tilde{\phi}\|[\|\phi\| + \|\phi_\xi\|] dt$$

$$\leq \frac{1}{16} \sup_{\tau \leq t \leq \tau + t_0} \|\tilde{\phi}\|^2 + C(M)t_0^2 \|\phi\|_{H^1(\mathbb{R})}^2, \tag{A.4}$$

where and in the sequel $C(M)$ is a generic positive constant depending on M and we have used the fact

$$\sup_{\tau \leq t \leq \tau + t_0} \|\phi\|_{L^\infty(\mathbb{R})} \leq C \sup_{\tau \leq t \leq \tau + t_0} \|\phi\|_{H^1(\mathbb{R})} \leq CM.$$

On the other hand, we have

$$\left| -4 \int_\tau^{\tau + t_0} \int_{\mathbb{R}} \frac{\tilde{\phi}(\phi_\xi + U_\xi)}{(u_- - u_+)^2} d\xi \int_{\mathbb{R}} a(U)U_\xi \phi d\xi dt \right|$$

$$\leq 4 \left| \int_\tau^{\tau + t_0} \int_{\mathbb{R}} \frac{\tilde{\phi}\phi_\xi}{(u_- - u_+)^2} d\xi \int_{\mathbb{R}} a(U)U_\xi \phi d\xi dt \right|$$

$$+ 4 \left| \int_\tau^{\tau + t_0} \int_{\mathbb{R}} \frac{\tilde{\phi}U_\xi}{(u_- - u_+)^2} d\xi \int_{\mathbb{R}} a(U)U_\xi \phi d\xi dt \right|$$

$$\leq C \left| \int_\tau^{\tau + t_0} \|\tilde{\phi}\| \|\phi_\xi\| \|\phi\|_{L^\infty(\mathbb{R})} dt \right|$$

$$+C\left|\int_\tau^{\tau+t_0}\left(\int_\mathbb{R}\frac{\tilde{\phi}U_\xi}{(u_- -u_+)^2}d\xi\right)^2+\left(\int_\mathbb{R}aU_\xi\phi d\xi\right)^2 dt\right|$$

$$\leq\frac{1}{16}\sup_{\tau\leq t\leq\tau+t_0}\|\tilde{\phi}\|^2+C(M)t_0^2\|\phi_\xi\|^2+C\int_\tau^{\tau+t_0}(\|\tilde{\phi}\|^2+\|\phi\|^2)dt$$

$$\leq\frac{1}{16}\sup_{\tau\leq t\leq\tau+t_0}\|\tilde{\phi}\|^2+C(M)\left(t_0+t_0^2\right)\|\phi\|_{H^1(\mathbb{R})}^2+Ct_0\sup_{\tau\leq t\leq\tau+t_0}\|\tilde{\phi}\|^2. \tag{A.5}$$

Substituting (A.4) and (A.5) into (A.3), and then choosing a suitably small $t_0=t_0(M)>0$, we have

$$\sup_{\tau\leq t\leq\tau+t_0}\|\tilde{\phi}\|^2+\int_\tau^{\tau+t_0}\|\tilde{\phi}_\xi\|^2 dt\leq 2M^2. \tag{A.6}$$

Multiplying the Eq. (A.2a) by $-\tilde{\phi}_{\xi\xi}$ and then integrating the resulting equation with respect to t and ξ over $[\tau,\tau+t_0]\times\mathbb{R}$, we can obtain

$$\frac{1}{2}\|\tilde{\phi}_\xi\|^2(\tau+t_0)+\int_\tau^{\tau+t_0}\|\tilde{\phi}_{\xi\xi}\|^2 dt$$

$$=\frac{1}{2}\|(\phi_\tau)_\xi\|^2+\int_\tau^{\tau+t_0}\int_\mathbb{R}\tilde{\phi}_{\xi\xi}[f(\phi+U)-f(U)]_\xi d\xi dt$$

$$+4\int_\tau^{\tau+t_0}\int_\mathbb{R}\frac{\tilde{\phi}_{\xi\xi}(\phi_\xi+U_\xi)}{(u_- -u_+)^2}d\xi\int_\mathbb{R}a(U)U_\xi\phi d\xi dt. \tag{A.7}$$

First, it holds that

$$\left|\int_\tau^{\tau+t_0}\int_\mathbb{R}\tilde{\phi}_{\xi\xi}[f(\phi+U)-f(U)]_\xi d\xi dt\right|$$

$$\leq\frac{1}{16}\int_\tau^{\tau+t_0}\|\tilde{\phi}_{\xi\xi}\|^2 dt+C\int_\tau^{\tau+t_0}\int_\mathbb{R}\left|[f(\phi+U)-f(U)]_\xi\right|^2 d\xi dt$$

$$\leq\frac{1}{16}\int_\tau^{\tau+t_0}\|\tilde{\phi}_{\xi\xi}\|^2 dt+C(M)\int_\tau^{\tau+t_0}\left(\|\phi\|^2+\|\phi_\xi\|^2\right)dt$$

$$\leq\frac{1}{16}\int_\tau^{\tau+t_0}\|\tilde{\phi}_{\xi\xi}\|^2 dt+C(M)t_0\sup_{\tau\leq t\leq\tau+t_0}\|\phi\|_{H^1(\mathbb{R})}^2. \tag{A.8}$$

Next, we can calculate

$$\left|4\int_\tau^{\tau+t_0}\int_\mathbb{R}\frac{\tilde{\phi}_{\xi\xi}\phi_\xi}{(u_- -u_+)^2}d\xi\int_\mathbb{R}a(U)U_\xi\phi d\xi dt\right|$$

$$\leq C\int_\tau^{\tau+t_0}\|\phi_\xi\|\|\tilde{\phi}_{\xi\xi}\|\|\phi\|_{L^\infty(\mathbb{R})}dt$$

$$\leq \frac{1}{16} \int_\tau^{\tau+t_0} \|\tilde{\phi}_{\xi\xi}\|^2 dt + C(M) \int_\tau^{\tau+t_0} \|\phi_\xi\|^2 dt$$

$$\leq \frac{1}{16} \int_\tau^{\tau+t_0} \|\tilde{\phi}_{\xi\xi}\|^2 dt + C(M) t_0 \sup_{\tau \leq t \leq \tau+t_0} \|\phi\|_{H^1(\mathbb{R})}^2, \qquad (A.9)$$

$$\left| 4 \int_\tau^{\tau+t_0} \int_\mathbb{R} \frac{\tilde{\phi}_{\xi\xi} U_\xi}{(u_- - u_+)^2} d\xi \int_\mathbb{R} a(U) U_\xi \phi \, d\xi dt \right|$$

$$\leq C \int_\tau^{\tau+t_0} \|\tilde{\phi}_{\xi\xi}\| \left| \int_\mathbb{R} a(U) U_\xi \phi \right| d\xi dt$$

$$\leq \frac{1}{16} \int_\tau^{\tau+t_0} \|\tilde{\phi}_{\xi\xi}\|^2 dt + C t_0 \sup_{\tau \leq t \leq \tau+t_0} \|\phi\|_{L^\infty(\mathbb{R})}^2$$

$$\leq \frac{1}{16} \int_\tau^{\tau+t_0} \|\tilde{\phi}_{\xi\xi}\|^2 dt + C t_0 \sup_{\tau \leq t \leq \tau+t_0} \|\phi\|_{H^1(\mathbb{R})}^2. \qquad (A.10)$$

Substituting (A.8), (A.9) and (A.10) into (A.7), and then choosing a suitably small $t_0 = t_0(M) > 0$, we have

$$\sup_{\tau \leq t \leq \tau+t_0} \|\tilde{\phi}_\xi\|^2 + \int_\tau^{\tau+t_0} \|\tilde{\phi}_{\xi\xi}\|^2 dt \leq 2M^2. \qquad (A.11)$$

By (A.6) and (A.11), we have

$$\sup_{\tau \leq t \leq \tau+t_0} \|\tilde{\phi}\|_{H^1(\mathbb{R})}^2 + \int_\tau^{\tau+t_0} \|\tilde{\phi}_\xi\|_{H^1(\mathbb{R})}^2 dt \leq 4M^2. \qquad (A.12)$$

Therefore, we proved $\tilde{\phi} \in Y_{\tau,t_0}^{2M}$. Next we prove that the above mapping $\mathcal{T} : Y_{\tau,t_0}^{2M} \to Y_{\tau,t_0}^{2M}$ is a contraction mapping in $C([\tau, \tau+t_0]; H^1(\mathbb{R}))$ for suitably small $t_0 > 0$. For this, for any $\phi_1, \phi_2 \in Y_{\tau,t_0}^{2M}$, denote $\Phi := \phi_2 - \phi_1$ and $\tilde{\Phi} := \mathcal{T}\phi_2 - \mathcal{T}\phi_1 = \tilde{\phi}_2 - \tilde{\phi}_1$. Then we have

$$\begin{cases} \tilde{\Phi}_t - \tilde{\Phi}_{\xi\xi} - s\tilde{\Phi}_\xi \\ = -[f(\phi_2+U) - f(\phi_1+U)]_\xi - \frac{4U_\xi}{(u_- - u_+)^2} \int_\mathbb{R} a(U) U_\xi \Phi d\xi \\ \qquad - \frac{4\phi_{2\xi}}{(u_- - u_+)^2} \int_\mathbb{R} a(U) U_\xi \Phi d\xi - \frac{4\Phi_\xi}{(u_- - u_+)^2} \int_\mathbb{R} a(U) U_\xi \phi_1 d\xi, \quad (A.13a) \\ \tilde{\Phi}(t=\tau, \xi) = 0. \qquad (A.13b) \end{cases}$$

Multiplying the Eq. (A.13a) by $\tilde{\Phi}$, and then integrating the resulting equation with respect to t and ξ over $[\tau, \tau+t_0] \times \mathbb{R}$, we have

$$\frac{1}{2}\|\tilde{\Phi}\|^2(\tau+t_0)+\int_\tau^{\tau+t_0}\|\tilde{\Phi}_\xi\|^2dt$$

$$=-\int_\tau^{\tau+t_0}\int_{\mathbb{R}}\tilde{\Phi}[f(\phi_2+U)-f(\phi_1+U)]_\xi d\xi dt$$

$$-4\int_\tau^{\tau+t_0}\int_{\mathbb{R}}\frac{\tilde{\Phi}U_\xi}{(u_--u_+)^2}d\xi\int_{\mathbb{R}}a(U)U_\xi\Phi d\xi dt$$

$$-4\int_\tau^{\tau+t_0}\int_{\mathbb{R}}\frac{\phi_{2\xi}\tilde{\Phi}}{(u_--u_+)^2}d\xi\int_{\mathbb{R}}a(U)U_\xi\Phi d\xi dt$$

$$-4\int_\tau^{\tau+t_0}\int_{\mathbb{R}}\frac{\Phi_\xi\tilde{\Phi}}{(u_--u_+)^2}d\xi\int_{\mathbb{R}}a(U)U_\xi\phi_1 d\xi dt. \tag{A.14}$$

We estimate the right-hand side of (A.14) terms by terms. First, we have

$$\left|-\int_\tau^{\tau+t_0}\int_{\mathbb{R}}\tilde{\Phi}[f(\phi_2+U)-f(\phi_1+U)]_\xi d\xi dt\right|$$

$$=\left|\int_\tau^{\tau+t_0}\int_{\mathbb{R}}\tilde{\Phi}_\xi[f(\phi_2+U)-f(\phi_1+U)]d\xi dt\right|$$

$$\leq C(M)\int_\tau^{\tau+t_0}\|\tilde{\Phi}_\xi\|\|\Phi\|dt$$

$$\leq \frac{1}{8}\int_\tau^{\tau+t_0}\|\tilde{\Phi}_\xi\|^2d\xi+C(M)t_0\sup_{\tau\leq t\leq\tau+t_0}\|\Phi\|^2. \tag{A.15}$$

Then we have

$$\left|-4\int_\tau^{\tau+t_0}\int_{\mathbb{R}}\frac{U_\xi\tilde{\Phi}}{(u_--u_+)^2}d\xi\int_{\mathbb{R}}a(U)U_\xi\Phi d\xi dt\right|$$

$$\leq C\left|\int_\tau^{\tau+t_0}\left(\int_{\mathbb{R}}U_\xi\tilde{\Phi}d\xi\right)^2+\left(\int_{\mathbb{R}}a(U)U_\xi\Phi d\xi\right)^2dt\right|$$

$$\leq Ct_0\left(\sup_{\tau\leq t\leq\tau+t_0}\|\tilde{\Phi}\|^2+\sup_{\tau\leq t\leq\tau+t_0}\|\Phi\|^2\right). \tag{A.16}$$

Finally, it holds that

$$\left|-4\int_\tau^{\tau+t_0}\int_{\mathbb{R}}\frac{\phi_{2\xi}\tilde{\Phi}}{(u_--u_+)^2}d\xi\int_{\mathbb{R}}a(U)U_\xi\Phi d\xi dt\right|$$

$$\leq C\left|\int_\tau^{\tau+t_0}\left(\int_{\mathbb{R}}\phi_{2\xi}\tilde{\Phi}d\xi\right)^2+\left(\int_{\mathbb{R}}a(U)U_\xi\Phi d\xi\right)^2dt\right|$$

$$\leq C(M)t_0\left(\sup_{\tau\leq t\leq\tau+t_0}\|\tilde{\Phi}\|^2+\sup_{\tau\leq t\leq\tau+t_0}\|\Phi\|^2\right), \tag{A.17}$$

$$\left| -4\int_\tau^{\tau+t_0}\int_\mathbb{R}\frac{\Phi_\xi\tilde{\Phi}}{(u_--u_+)^2}d\xi\int_\mathbb{R}a(U)U_\xi\phi_1d\xi dt\right|$$

$$\leq C(M)\int_\tau^{\tau+t_0}\|\Phi_\xi\|\|\tilde{\Phi}\|dt$$

$$\leq\frac{1}{8}\sup_{\tau\leq t\leq\tau+t_0}\|\tilde{\Phi}\|^2+C(M)t_0^2\sup_{\tau\leq t\leq\tau+t_0}\|\Phi_\xi\|^2. \tag{A.18}$$

Substituting (A.15)-(A.18) into (A.14) and then taking t_0 smaller than before if needed, we have

$$\sup_{\tau\leq t\leq\tau+t_0}\|\tilde{\Phi}\|^2\leq\frac{1}{3}\sup_{\tau\leq t\leq\tau+t_0}\|\Phi\|_{H^1(\mathbb{R})}^2. \tag{A.19}$$

Multiplying the Eq. (A.13a) by $-\tilde{\Phi}_{\xi\xi}$, and then integrating the resulting equation with respect to t and ξ over $[\tau,\tau+t_0]\times\mathbb{R}$, we can get

$$\frac{1}{2}\|\tilde{\Phi}_\xi\|^2(\tau+t_0)+\int_\tau^{\tau+t_0}\|\tilde{\Phi}_{\xi\xi}\|^2dt$$

$$=\int_\tau^{\tau+t_0}\int_\mathbb{R}\tilde{\Phi}_{\xi\xi}[f(\phi_2+U)-f(\phi_1+U)]_\xi d\xi dt$$

$$+4\int_\tau^{\tau+t_0}\int_\mathbb{R}\frac{\tilde{\Phi}_{\xi\xi}U_\xi}{(u_--u_+)^2}d\xi\int_\mathbb{R}a(U)U_\xi\Phi d\xi dt$$

$$+4\int_\tau^{\tau+t_0}\int_\mathbb{R}\frac{\phi_{2\xi}\tilde{\Phi}_{\xi\xi}}{(u_--u_+)^2}d\xi\int_\mathbb{R}a(U)U_\xi\Phi d\xi dt$$

$$+4\int_\tau^{\tau+t_0}\int_\mathbb{R}\frac{\Phi_\xi\tilde{\Phi}_{\xi\xi}}{(u_--u_+)^2}d\xi\int_\mathbb{R}a(U)U_\xi\phi_1d\xi dt. \tag{A.20}$$

We need to estimate the right-hand side of (A.20). First, it holds that

$$\left|\int_\tau^{\tau+t_0}\int_\mathbb{R}\tilde{\Phi}_{\xi\xi}[f(\phi_2+U)-f(\phi_1+U)]_\xi d\xi dt\right|$$

$$\leq\frac{1}{16}\int_\tau^{\tau+t_0}\|\tilde{\Phi}_{\xi\xi}\|^2dt+C(M)\int_\tau^{\tau+t_0}(\|\Phi\|^2+\|\Phi_\xi\|^2)dt$$

$$\leq\frac{1}{16}\int_\tau^{\tau+t_0}\|\tilde{\Phi}_{\xi\xi}\|^2dt+C(M)t_0\sup_{\tau\leq t\leq\tau+t_0}(\|\Phi\|^2+\|\Phi_\xi\|^2). \tag{A.21}$$

Then we can compute that

$$\left|4\int_\tau^{\tau+t_0}\int_\mathbb{R}\frac{\tilde{\Phi}_{\xi\xi}U_\xi}{(u_--u_+)^2}d\xi\int_\mathbb{R}a(U)U_\xi\Phi d\xi dt\right|$$

$$\leq \frac{1}{16} \sup_{\tau \leq t \leq \tau+t_0} \|\tilde{\Phi}_{\xi\xi}\|^2 + Ct_0^2 \sup_{\tau \leq t \leq \tau+t_0} \|\Phi\|_{L^\infty(\mathbb{R})}^2$$

$$\leq \frac{1}{16} \sup_{\tau \leq t \leq \tau+t_0} \|\tilde{\Phi}_{\xi\xi}\|^2 + Ct_0^2 \sup_{\tau \leq t \leq \tau+t_0} \|\Phi\|_{H^1(\mathbb{R})}^2, \tag{A.22}$$

$$\left| 4 \int_\tau^{\tau+t_0} \int_\mathbb{R} \frac{\phi_{2\xi}\tilde{\Phi}_{\xi\xi}}{(u_- - u_+)^2} d\xi \int_\mathbb{R} a(U)U_\xi \Phi d\xi dt \right|$$

$$\leq C \int_\tau^{\tau+t_0} \|\tilde{\Phi}_{\xi\xi}\| \|\phi_{2\xi}\| \|\Phi\|_{L^\infty(\mathbb{R})} dt$$

$$\leq \frac{1}{16} \int_\tau^{\tau+t_0} \|\tilde{\Phi}_{\xi\xi}\|^2 dt + C(M)t_0 \sup_{\tau \leq t \leq \tau+t_0} \|\Phi\|_{H^1(\mathbb{R})}^2, \tag{A.23}$$

$$\left| 4 \int_\tau^{\tau+t_0} \int_\mathbb{R} \frac{\Phi_\xi \tilde{\Phi}_{\xi\xi}}{(u_- - u_+)^2} d\xi \int_\mathbb{R} a(U)U_\xi \phi_1 d\xi dt \right|$$

$$\leq C \int_\tau^{\tau+t_0} \|\Phi_\xi\| \|\tilde{\Phi}_{\xi\xi}\| \|\phi_1\|_{L^\infty(\mathbb{R})} dt$$

$$\leq \frac{1}{16} \int_\tau^{\tau+t_0} \|\tilde{\Phi}_{\xi\xi}\|^2 dt + C(M)t_0 \sup_{\tau \leq t \leq \tau+t_0} \|\Phi\|_{H^1(\mathbb{R})}^2. \tag{A.24}$$

Substituting (A.21)-(A.24) into (A.20) and then taking t_0 suitably smaller than before if needed, we have

$$\sup_{\tau \leq t \leq \tau+t_0} \|\tilde{\Phi}_\xi\|^2 \leq \frac{1}{3} \sup_{\tau \leq t \leq \tau+t_0} \|\Phi\|_{H^1(\mathbb{R})}^2. \tag{A.25}$$

By (A.19) and (A.25), we have

$$\sup_{\tau \leq t \leq \tau+t_0} \|\tilde{\Phi}\|_{H^1(\mathbb{R})}^2 \leq \frac{2}{3} \sup_{\tau \leq t \leq \tau+t_0} \|\Phi\|_{H^1(\mathbb{R})}^2. \tag{A.26}$$

Hence, the mapping $\mathcal{T}: Y_{\tau,t_0}^{2M} \to Y_{\tau,t_0}^{2M}$ is a contraction mapping in $C([\tau, \tau+t_0]; H^1(\mathbb{R}))$ for suitably small $t_0 = t_0(M) > 0$. Therefore, there exist a suitably small $t_0 = t_0(M) > 0$ and a unique fixed point $\phi \in Y_{\tau,t_0}^{2M} \subset C([\tau, \tau+t_0]; H^1(\mathbb{R}))$ for the mapping $\mathcal{T}$, that is

$$\phi(t, \xi) = \mathcal{T}\phi(t, \xi).$$

Then we finish the proof of Proposition 2.1. $\square$

Acknowledgments

References

[1] P. Blochas and J. Cheng, *Viscous destabilization for large shocks of conservation laws*, arXiv2501.01537, 2025.

[2] H. Freistühler and D. Serre, L^1-*stability of shock waves in scalar viscous conservation laws*, Commun. Pure Appl. Math. 51 (1998), 291–301.

[3] J. Goodman, *Nonlinear asymptotic stability of viscous shock profiles for conservation laws*, Arch. Ration. Mech. Anal. 95 (1986), 325–344.

[4] J. Goodman, *Stability of viscous scalar shock fronts in several dimensions*, Trans. Amer. Math. Soc. 311 (1989), 683–695.

[5] E. Hopf, *The partial differential equations $u_t + uu_x = \mu u_{xx}$*, Commun. Pure. Appl. Math. 3 (1950), 201–230.

[6] F. M. Huang, Y. Wang, and J. Zhang, *Time-asymptotic stability of composite waves of degenerate Oleinik shock and rarefaction for non-convex conservation laws*, Math. Ann. 392 (2025), 1–46.

[7] J. Humpherys, G. Lyng, and K. Zumbrun, *Multidimensional stability of large-amplitude Navier-Stokes shocks*, Arch. Ration. Mech. Anal. 226 (2017), 923–973.

[8] A. M. Il'in and O. A. Oleinik, *Asymptotic behavior of solution of the Cauchy problem for some quasilinear equations for large values of the time*, Mat. Sb. (N.S.) 51(93) (1960), 191–216.

[9] M.-J. Kang, L^2 *-type contraction for shocks of scalar viscous conservation laws with strictly convex flux*, J. Math. Pures Appl. 145 (2021), 1–43.

[10] M.-J. Kang and H. Oh, L^2 *decay for large perturbations of viscous shocks for multi-D Burgers equation*, arXiv2430.08445, 2024.

[11] M.-J. Kang and A. Vasseur, *Criteria on contractions for entropic discontinuities of systems of conservation laws*, Arch. Ration. Mech. Anal. 222(1) (2016), 343–391.

[12] M.-J. Kang and A. Vasseur, L^2-*contraction for shock waves of scalar viscous conservation laws*, Ann. Inst. H. Poincare C Anal. Non Lineaire 34 (1) (2017), 139–156.

[13] M.-J. Kang and A. Vasseur, *Contraction property for large perturbations of shocks of the barotropic Navier-Stokes system*, J. Eur. Math. Soc. 23(2) (2020), 585–638.

[14] M.-J. Kang, A. Vasseur, and Y. Wang, L^2-*contraction of large planar shock waves for multi-dimensional scalar viscous conservation laws*, J. Differential Equations 267(5) (2019), 2737–2791.

[15] M.-J. Kang, A. Vasseur, and Y. Wang, *Time-asymptotic stability of composite waves of viscous shock and rarefaction for barotropic Navier-Stokes equations*, Adv. Math. 419 (2023), 108963.

[16] M.-J. Kang, A. Vasseur, and Y. Wang, *Time-Asymptotic stability of generic Riemann solutions for compressible Navier-Stokes-Fourier equations*, Arch. Ration. Mech. Anal. 249 (2025), 42.

[17] S. Kawashima and A. Matsumura, *Asymptotic stability of traveling wave solutions of systems for one-dimensional gas motion*, Comm. Math. Phys. 101 (1985), 97–127.

[18] C. E. Kenig and F. Merle, *Asymptotic stability and Liouville theorem for scalar viscous conservation laws in cylinders*, Commun. Pure Appl. Math. 59 (2006), 769–796.

[19] S. N. Kružkov, *First order quasilinear equations with several independent variables*, Mat. Sb. (N.S.) 81(123) (1970), 228–255.

[20] T.-P. Liu, *Nonlinear stability of shock waves for viscous conservation laws*, Bull. Amer. Math. Soc. (N.S.) 12(2) (1985), 233–236.

[21] T.-P. Liu and Y. N. Zeng, *Shock Waves in Conservation Laws With Physical Viscosity*, in: Memoirs of the American Mathematical Society, Vol. 234(1105), 2015.

[22] C. Mascia and K. Zumbrun, *Stability of small-amplitude shock profiles of symmetric hyperbolic-parabolic systems*, Commun. Pure Appl. Math. 57 (2004), 841–876.

[23] A. Matsumura and K. Nishihara, *On the stability of traveling wave solutions of a one-dimensional model system for compressible viscous gas*, Japan J. Appl. Math. 2 (1985), 17–25.

[24] K. Nishihara, *A note on the stability of traveling wave solutions of the Burgers' equation*, Japan J. Appl. Math. 2(1) (1985), 27–35.

[25] D. Sattinger, *On the stability of waves of nonlinear parabolic systems*, Adv. Math. 22(3) (1976), 312–355.

[26] D. Serre, L^1-*stability of nonlinear waves in scalar conservation laws*, in: Handbook of Differential Equations: Evolutionary Equations, Vol. 1, North-Holland, 473–553, 2002.

[27] A. Szepessy and Z.-P. Xin, *Nonlinear stability of viscous shock waves*, Arch. Ration. Mech. Anal. 122 (1993), 53–104.

[28] T. Wang and Y. Wang, *Nonlinear stability of planar viscous shock wave to three-dimensional compressible Navier-Stokes equations*, J. Eur. Math. Soc., (2024), doi: 10.4171/JEMS/1486.

[29] K. Zumbrun, *Planar stability criteria for viscous shock waves of systems with real viscosity*, in: Hyperbolic Systems of Balance Laws. Lecture Notes in Mathematics, Vol. 1911, Springer, 229–326, 2007.