

Classical Solutions to a Model for Heat Generation During Acoustic Wave Propagation in a Standard Linear Solid

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Abstract. In an open bounded real interval, this manuscript studies the evolution system

$$\begin{cases} u_{ttt} + \alpha u_{tt} = (\gamma(\Theta)u_{xt})_x + (\hat{\gamma}(\Theta)u_x)_{x'} \\ \Theta_t = D\Theta_{xx} + \Gamma(\Theta)u_{xt}^2, \end{cases}$$

which arises as a model for the generation of heat during propagation of acoustic waves in a standard linear solid.

A statement in local existence and uniqueness of classical solutions is derived for arbitrary $D > 0$ and $\alpha \geq 0$, for sufficiently smooth $\gamma, \hat{\gamma}$ and Γ with $\gamma > 0, \hat{\gamma} > 0$ and $\Gamma \geq 0$ on $[0, \infty)$, and for all suitably regular initial data of arbitrary size.

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1 Introduction

This manuscript is concerned with the initial-boundary value problem

$$\begin{cases} u_{ttt} + \alpha u_{tt} = (\gamma(\Theta)u_{xt})_x + (\hat{\gamma}(\Theta)u_x)_{x'}, & x \in \Omega, \quad t > 0, \\ \Theta_t = D\Theta_{xx} + \Gamma(\Theta)u_{xt}^2, & x \in \Omega, \quad t > 0, \\ u_x = 0, \quad \Theta_x = 0, & x \in \partial\Omega, \quad t > 0, \\ (u, u_t, u_{tt}, \Theta)(x, 0) = (u_0, u_{0t}, u_{0tt}, \Theta_0)(x), & x \in \Omega, \end{cases} \quad (1.1)$$

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which arises as a simplified model for heat generation during acoustic wave propagation in so-called standard linear solids. Indeed, assuming a one-dimensional material to be of Zener type suggests to consider the displacement variable u to be determined by the third-order hyperbolic equation in (1.1), where recent experimental observations suggest certain dependencies on the temperature Θ in the elastic parameters $\hat{\gamma}$ and γ related to stiffness and viscosity near relevant operating points [24]. In slight modification of a more complex model, (1.1) moreover assumes elastic behavior to be dominant as compared to mechanical losses, so that additional and temporally nonlocal contributions to the source in the heat subsystem of (1.1) can be neglected. For more details on the modeling background, we may refer to [9, 17]; we note that of predominant application relevance in purely thermomechanical settings seem scenarios in which $\Gamma \sim \gamma$, while including interaction with electric fields, such as of interest in the modeling of heat generation in piezoceramics, leads to choices with $\hat{\gamma}(\Theta) = a\gamma(\Theta) + d$ with some $a > 0$ and $d > 0$ (see, e.g. [22, 44] for corresponding considerations detailed for a second-order relative of (1.1)).

In contrast to classical models for thermoviscoelastic evolution in Kelvin-Voigt materials which have undergone a thorough mathematical analysis over the past decades [4, 8, 18, 33, 39, 40], third-order models of the form in (1.1) seem to have become objects of study only in more recent literature. Despite the potential of the Zener modeling approach to appropriately capture key properties of standard linear solids such as instantaneous elastic responses and stress relaxation, a corresponding description by means of Moore-Gibson-Thompson equations such as in (1.1) to date seems to have mainly concentrated on linear cases (cf., e.g. [1, 3, 21, 42]).

The mathematical analysis of acoustic problems containing Moore-Gibson-Thompson type subsystems thus seems to have focused on models for wave propagation in non-solid materials in which u can be chosen to represent a scalar pressure variable so far, and in which interaction with temperature fields, if of relevance at all, commonly leads to different types of couplings. Accordingly, analytical studies have been concerned with aspects of stability and large time decay in linear Moore-Gibson-Thompson problems in various special frameworks both in bounded n -dimensional domains [11, 20, 26, 31] and the entire space \mathbb{R}^n [11, 12], partially also including certain memory terms [2, 19, 29, 30]; studies on nonlinear relatives have mainly addressed semilinear situations that involve various types of source terms depending either on the solution itself or some of its derivatives, focusing either on questions of local and global solvability [27, 28, 38, 41], or on the occurrence of blow-up phenomena [11, 13, 14, 34, 37].

Quasilinear couplings of acoustic systems to temperature fields such as in (1.1), however, seem to have been examined mainly in frameworks of second-order models for the evolution of the respective mechanical parts. For the so-called Westervelt-Pennes system which can be viewed as a second-order relative of (1.1) that involves a slightly less destabilizing heat source but additionally contains a second-order nonlinear forcing term in the corresponding wave part, results inter alia on global small-data solutions have been achieved in the literature of the past five years [5, 6, 10, 35, 36]; also recently, a different class of examples for second-order modifications of (1.1) which involve quasilinear interaction with a temperature variable, actually addressing Kelvin-Voigt type simplifications of (1.1), has been examined with regard to issues from basic existence and nonexistence theory in various contributions [15, 16, 22, 32, 43, 45]. Only very recently, in [7] a similar type of quasilinear coupling has been studied in the context of a third-order wave equation, and a statement on local existence of certain small-data solutions for the resulting so-called Moore-Gibson-Thompson-Pennes system has been derived there.

Main results. For the quasilinear Zener-type model (1.1), however, available knowledge from mathematical analysis so far seems to reduce to a result on local existence in a suitably generalized framework of solvability. In particular, in [17] the following statement in this regard has been achieved, particularly covering initial data of arbitrary size unlike the approach in [7]; in formulating this and throughout the sequel, for a fixed bounded interval $\Omega \subset \mathbb{R}$ and arbitrary $p \in (1, \infty]$ we let

$$W_N^{2,p}(\Omega) := \left\{ \varphi \in W^{2,p}(\Omega) \mid \frac{\partial \varphi}{\partial \nu} = 0 \text{ on } \partial\Omega \right\}.$$

Proposition 1.1. *Let $\Omega \subset \mathbb{R}$ be an open bounded interval, let $D > 0$ and $\alpha \geq 0$, let $\gamma \in C^2([0, \infty))$, $\hat{\gamma} \in C^2([0, \infty))$ and $\Gamma \in C^1([0, \infty))$ be such that $\gamma > 0$, $\hat{\gamma} > 0$ and $\Gamma \geq 0$ on $[0, \infty)$, and let*

$$u_0 \in W_N^{2,2}(\Omega), \quad u_{0t} \in W_N^{2,2}(\Omega), \quad u_{0tt} \in W^{1,2}(\Omega), \quad \Theta_0 \in W_N^{2,\infty}(\Omega) \quad (1.2)$$

satisfy $\Theta_0 \geq 0$ in Ω . Then there exist $T_{\max} \in (0, \infty]$ and uniquely determined functions u and Θ with

$$\begin{cases} u \in C^0([0, T_{\max}); W_N^{2,2}(\Omega)), \\ u_t \in C^0([0, T_{\max}); C^1(\bar{\Omega})) \cap L_{loc}^\infty([0, T_{\max}); W_N^{2,2}(\Omega)), \\ u_{tt} \in C^0(\bar{\Omega} \times [0, T_{\max})) \cap L_{loc}^\infty((0, T_{\max}); W^{1,2}(\Omega)), \\ \Theta \in C^0([0, T_{\max}); C^1(\bar{\Omega})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})) \cap W_{loc}^{1,\infty}(\bar{\Omega} \times [0, T_{\max})) \end{cases} \quad (1.3)$$

such that $\Theta \geq 0$ in $\Omega \times (0, T_{\max})$, that if $T_{\max} < \infty$, then

$$\limsup_{t \nearrow T_{\max}} \{ \|u_t(\cdot, t)\|_{W^{2,2}(\Omega)} + \|u_{tt}(\cdot, t)\|_{W^{1,2}(\Omega)} + \|\Theta(\cdot, t)\|_{W^{2,\infty}(\Omega)} \} = \infty, \tag{1.4}$$

and that (u, Θ) forms a strong solution of (1.1) in the sense that $u(\cdot, 0) = u_0$ and $u_t(\cdot, 0) = u_{0t}$ in Ω , that

$$\begin{aligned} & - \int_0^T \int_{\Omega} u_{tt} \varphi_t - \int_{\Omega} u_{0tt} \varphi(\cdot, 0) + \alpha \int_0^T \int_{\Omega} u_{tt} \varphi \\ & = - \int_0^T \int_{\Omega} \gamma(\Theta) u_{xt} \varphi_x - \int_0^T \int_{\Omega} \hat{\gamma}(\Theta) u_x \varphi_x \end{aligned} \tag{1.5}$$

for each $\varphi \in C_0^\infty(\bar{\Omega} \times [0, T))$, and that $\Theta_t = D\Theta_{xx} + \Gamma(\Theta)u_{xt}^2$ in $\Omega \times (0, T)$ as well as $\Theta_x = 0$ on $\partial\Omega \times (0, T)$.

Due to an apparent lack of appropriate smoothing in the hyperbolic subsystem of (1.1), we do not expect assumptions on the initial data as mild as those in (1.2) to entail solution regularity substantially beyond that described by (1.3); in particular, it is to be suspected that the hypotheses in (1.2) are insufficient to generate classical solutions.

The purpose of the present manuscript now consists in making sure that suitably enhanced regularity requirements on the initial data as well as the parameter functions in (1.1) indeed ensure the existence actually also of classical solutions. This will be addressed by examining short-time evolution properties of functionals of the form

$$y(t) := \frac{1}{2} \int_{\Omega} u_{xxtt}^2 + \frac{1}{2} \int_{\Omega} \gamma(\Theta) u_{xxxt}^2 + \int_{\Omega} \hat{\gamma}(\Theta) u_{xxx} u_{xxxt} + \frac{B}{2} \int_{\Omega} u_{xxx}^2 \tag{1.6}$$

with suitably large $B > 0$. As to be detailed in Lemma 3.3, namely, on the basis of some lower-order regularity properties, gathered in the course of a local-in-time analysis in [17] (see Lemma 2.2 below), suitably regularized variants of these quantities can be seen to grow in a suitably controllable extent on small timescales. This observation will form the core of our reasoning toward the following main result of this manuscript, asserting local classical solvability along with a handy extensibility criterion.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}$ be an open bounded interval, let $D > 0$ and $\alpha \geq 0$, and suppose that*

$$\begin{cases} \gamma \in C^3([0, \infty)) & \text{is such that } \gamma > 0 \text{ on } [0, \infty), \\ \hat{\gamma} \in C^3([0, \infty)) & \text{is such that } \hat{\gamma} > 0 \text{ on } [0, \infty), \\ \Gamma \in C^1([0, \infty)) & \text{is such that } \Gamma \geq 0 \text{ on } [0, \infty), \end{cases} \tag{1.7}$$

and that

$$\begin{cases} u_0 \in W_N^{2,2}(\Omega) \cap W^{3,2}(\Omega), \\ u_{0t} \in W_N^{2,2}(\Omega) \cap W^{3,2}(\Omega), \\ u_{0tt} \in W_N^{2,2}(\Omega) \cap W^{1,2}(\Omega), \\ \Theta_0 \in W_N^{2,\infty}(\Omega) \text{ is such that } \Theta_0 \geq 0 \text{ in } \Omega. \end{cases} \quad (1.8)$$

Then there exist $T_{\max} \in (0, \infty]$ as well as uniquely determined functions

$$\begin{cases} u \in C^0([0, T_{\max}); C^2(\overline{\Omega})) \cap L^\infty((0, T_{\max}); W^{3,2}(\Omega)), \\ \Theta \in C^0([0, T_{\max}); C^1(\overline{\Omega})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})) \cap W_{loc}^{1,\infty}(\overline{\Omega} \times [0, T_{\max})) \end{cases} \quad (1.9)$$

such that

$$u_t \in C^0([0, T_{\max}); C^2(\overline{\Omega})) \cap L^\infty((0, T_{\max}); W^{3,2}(\Omega)), \quad (1.10)$$

$$u_{tt} \in C^0([0, T_{\max}); C^1(\overline{\Omega})) \cap L^\infty((0, T_{\max}); W^{2,2}(\Omega)) \quad (1.11)$$

as well as

$$u_{ttt} \in C^0(\overline{\Omega} \times [0, T_{\max})) \cap L^\infty((0, T_{\max}); W^{1,2}(\Omega)), \quad (1.12)$$

that (u, Θ) solves (1.1) in the classical sense in $\Omega \times (0, T_{\max})$, and that if $T_{\max} < \infty$, then

$$\limsup_{t \nearrow T_{\max}} \{ \|u_t(\cdot, t)\|_{W^{3,2}(\Omega)} + \|u_{tt}(\cdot, t)\|_{W^{2,2}(\Omega)} + \|\Theta(\cdot, t)\|_{W^{2,\infty}(\Omega)} \} = \infty. \quad (1.13)$$

2 Preliminaries

Let us briefly recall and adapt from [17] some steps toward the design of a suitable parabolic regularization of (1.1), and import some preliminary estimates that have been gained in [17] actually under less restrictive assumptions than those made here.

Specifically, here and below we consider the open interval $\Omega \subset \mathbb{R}$ as well as the constants $D > 0$ and $\alpha \geq 0$ fixed, let $\gamma, \hat{\gamma}$ and Γ satisfy (1.7), and suppose that the initial data u_0, u_{0t}, u_{0tt} and Θ_0 are such that (1.2) holds. Unless otherwise stated, we shall additionally assume that

$$\int_{\Omega} u_0 = \int_{\Omega} u_{0t} = \int_{\Omega} u_{0tt} = 0. \quad (2.1)$$

Then using standard smoothing procedures yields families $(\hat{\gamma}_\varepsilon)_{\varepsilon \in (0,1)} \subset C^\infty([0, \infty))$, $(\gamma_\varepsilon)_{\varepsilon \in (0,1)} \subset C^\infty([0, \infty))$, $(\hat{\gamma}_\varepsilon)_{\varepsilon \in (0,1)} \subset C^\infty([0, \infty))$ and $(\Gamma_\varepsilon)_{\varepsilon \in (0,1)} \subset C^\infty([0, \infty))$ as well

as approximations $(u_{0\varepsilon})_{\varepsilon \in (0,1)} \subset C^\infty(\overline{\Omega})$, $(v_{0\varepsilon})_{\varepsilon \in (0,1)} \subset C^\infty(\overline{\Omega})$, $(w_{0\varepsilon})_{\varepsilon \in (0,1)} \subset C^\infty(\overline{\Omega})$, $(\Theta_{0\varepsilon})_{\varepsilon \in (0,1)} \subset C^\infty(\overline{\Omega})$ of the initial distributions $u_0, v_0 := u_{0t}$ and $w_0 := u_{0tt}$ of the displacement variable and its first and second order time derivatives, as well as the initial temperature Θ_0 such that

$$\gamma_\varepsilon > 0, \quad \widehat{\gamma}_\varepsilon > 0, \quad \Gamma_\varepsilon \geq 0 \quad \text{on } [0, \infty), \quad \forall \varepsilon \in (0, 1), \quad (2.2)$$

that $u_{0\varepsilon x}, v_{0\varepsilon x}, w_{0\varepsilon x}$ and $\Theta_{0\varepsilon x}$ have compact support in Ω and $\Theta_{0\varepsilon} \geq 0$ with

$$\int_{\Omega} u_{0\varepsilon} = \int_{\Omega} v_{0\varepsilon} = \int_{\Omega} w_{0\varepsilon} = 0, \quad \forall \varepsilon \in (0, 1), \quad (2.3)$$

and that as $\varepsilon \searrow 0$ we have

$$\begin{aligned} \gamma_\varepsilon &\rightarrow \gamma \quad \text{in } C_{loc}^3([0, \infty)), \\ \widehat{\gamma}_\varepsilon &\rightarrow \widehat{\gamma} \quad \text{in } C_{loc}^3([0, \infty)), \\ \Gamma_\varepsilon &\rightarrow \Gamma \quad \text{in } C_{loc}^1([0, \infty)) \end{aligned} \quad (2.4)$$

and

$$\begin{cases} u_{0\varepsilon} \rightarrow u_0 & \text{in } W^{3,2}(\Omega), \\ v_{0\varepsilon} \rightarrow u_{0t} & \text{in } W^{3,2}(\Omega), \\ w_{0\varepsilon} \rightarrow u_{0tt} & \text{in } W^{2,2}(\Omega), \\ \Theta_{0\varepsilon} \xrightarrow{*} \Theta_0 & \text{in } W^{2,\infty}(\Omega) \end{cases} \quad (2.5)$$

as well as

$$\sqrt{\varepsilon} u_{0\varepsilon xxx} \rightarrow 0 \quad \text{in } L^2(\Omega). \quad (2.6)$$

As seen in [17, Lemma 3.1], the parabolic the regularization of (1.1), as given for $\varepsilon \in (0, 1)$ by

$$\begin{cases} w_{\varepsilon t} = \varepsilon w_{\varepsilon xx} + (\gamma_\varepsilon(\Theta_\varepsilon)v_{\varepsilon x})_x + (\widehat{\gamma}_\varepsilon(\Theta_\varepsilon)u_{\varepsilon x})_x - \alpha w_\varepsilon, & x \in \Omega, \quad t > 0, & (2.7a) \\ v_{\varepsilon t} = \varepsilon v_{\varepsilon xx} + w_\varepsilon, & x \in \Omega, \quad t > 0, & (2.7b) \\ u_{\varepsilon t} = \varepsilon u_{\varepsilon xx} + v_\varepsilon, & x \in \Omega, \quad t > 0, & (2.7c) \\ \Theta_{\varepsilon t} = D\Theta_{\varepsilon xx} + \Gamma_\varepsilon(\Theta_\varepsilon)v_{\varepsilon x}^2, & x \in \Omega, \quad t > 0, & (2.7d) \\ w_{\varepsilon x} = v_{\varepsilon x} = u_{\varepsilon x} = \Theta_{\varepsilon x} = 0, & x \in \partial\Omega, \quad t > 0, & (2.7e) \\ (w_\varepsilon, v_\varepsilon, u_\varepsilon, \Theta_\varepsilon)(x, 0) = (w_{0\varepsilon}, v_{0\varepsilon}, u_{0\varepsilon}, \Theta_{0\varepsilon})(x), & x \in \Omega, & (2.7f) \end{cases}$$

then admits local smooth solutions in the following sense.

Lemma 2.1. *Let $\varepsilon \in (0,1)$. Then there exist $T_{\max,\varepsilon} \in (0,\infty]$ and*

$$\begin{cases} w_\varepsilon \in C^{2,1}(\overline{\Omega} \times [0, T_{\max,\varepsilon})) \cap C^\infty(\overline{\Omega} \times (0, T_{\max,\varepsilon})), \\ v_\varepsilon \in C^{2,1}(\overline{\Omega} \times [0, T_{\max,\varepsilon})) \cap C^\infty(\overline{\Omega} \times (0, T_{\max,\varepsilon})), \\ u_\varepsilon \in C^{2,1}(\overline{\Omega} \times [0, T_{\max,\varepsilon})) \cap C^\infty(\overline{\Omega} \times (0, T_{\max,\varepsilon})), \\ \Theta_\varepsilon \in C^{2,1}(\overline{\Omega} \times [0, T_{\max,\varepsilon})) \cap C^\infty(\overline{\Omega} \times (0, T_{\max,\varepsilon})) \end{cases} \quad (2.8)$$

such that $\Theta_\varepsilon \geq 0$ in $\Omega \times (0, T_{\max,\varepsilon})$, that $(w_\varepsilon, v_\varepsilon, u_\varepsilon, \Theta_\varepsilon)$ solves (2.7) classically in $\Omega \times (0, T_{\max,\varepsilon})$, and that if $T_{\max,\varepsilon} < \infty$, then

$$\limsup_{t \nearrow T_{\max,\varepsilon}} \{ \|w_\varepsilon(\cdot, t)\|_{W^{1,2}(\Omega)} + \|v_\varepsilon(\cdot, t)\|_{W^{1,2}(\Omega)} + \|u_\varepsilon(\cdot, t)\|_{W^{1,2}(\Omega)} + \|\Theta_\varepsilon(\cdot, t)\|_{W^{1,2}(\Omega)} \} = \infty. \quad (2.9)$$

Moreover,

$$\int_\Omega w_\varepsilon(\cdot, t) = \int_\Omega v_\varepsilon(\cdot, t) = \int_\Omega u_\varepsilon(\cdot, t) = 0, \quad \forall t \in (0, T_{\max,\varepsilon}). \quad (2.10)$$

Now under assumptions actually less restrictive than those in (1.7), (1.8), (2.2), (2.4)-(2.6), in [17, Corollary 3.12] these solutions have been found to enjoy the following basic regularity properties on appropriately short time intervals.

Lemma 2.2. *There exist $\varepsilon_\star \in (0,1)$, $T_\star > 0$ and $C > 0$ such that $T_{\max,\varepsilon} \geq T_\star$ for all $\varepsilon \in (0, \varepsilon_\star)$, and that*

$$\|w_\varepsilon(\cdot, t)\|_{W^{1,2}(\Omega)} \leq C, \quad \forall t \in (0, T_\star), \quad \varepsilon \in (0, \varepsilon_\star), \quad (2.11)$$

$$\|v_\varepsilon(\cdot, t)\|_{W^{2,2}(\Omega)} \leq C, \quad \forall t \in (0, T_\star), \quad \varepsilon \in (0, \varepsilon_\star), \quad (2.12)$$

$$\|u_\varepsilon(\cdot, t)\|_{W^{2,2}(\Omega)} \leq C, \quad \forall t \in (0, T_\star), \quad \varepsilon \in (0, \varepsilon_\star), \quad (2.13)$$

$$\|\Theta_{\varepsilon t}(\cdot, t)\|_{L^\infty(\Omega)} \leq C, \quad \forall t \in (0, T_\star), \quad \varepsilon \in (0, \varepsilon_\star), \quad (2.14)$$

$$\|\Theta_\varepsilon(\cdot, t)\|_{W^{2,\infty}(\Omega)} \leq C, \quad \forall t \in (0, T_\star), \quad \varepsilon \in (0, \varepsilon_\star). \quad (2.15)$$

As a consequence of accordingly implied compactness properties, these approximate solutions approach strong solutions of the original problem (1.1). With regard to the topological framework, the following statement in this regard col-

lects from a corresponding more comprehensive result in [17, Lemma 3.14] only what will be needed later on.

Lemma 2.3. *Let ε_\star and T_\star be as in Lemma 2.2. Then there exist $(\varepsilon_j)_{j \in \mathbb{N}} \subset (0, \varepsilon_\star)$ and a strong solution (u, Θ) of (1.1), in the sense specified Proposition 1.1 such that $\varepsilon_j \searrow 0$ as $j \rightarrow \infty$ and*

$$u_\varepsilon \rightarrow u \quad \text{in } C^0([0, T_\star]; C^1(\overline{\Omega})), \tag{2.16}$$

$$v_\varepsilon \rightarrow u_t \quad \text{in } C^0([0, T_\star]; C^1(\overline{\Omega})), \tag{2.17}$$

$$w_\varepsilon \rightarrow u_{tt} \quad \text{in } C^0(\overline{\Omega} \times [0, T_\star]), \tag{2.18}$$

$$\Theta_\varepsilon \rightarrow \Theta \quad \text{in } C^0(\overline{\Omega} \times [0, T_\star]) \tag{2.19}$$

as $\varepsilon = \varepsilon_j \searrow 0$.

3 Short-time analysis of a higher-order quasi-energy functional

A first ε -independent estimate beyond those from Lemma 2.2 makes use of the diffusion mechanism in the heat subsystem of (2.7), and does actually not rely on the sharpened assumptions made e.g. in (1.8).

Lemma 3.1. *There exists $C > 0$ such that if we let ε_\star and T_\star be as in Lemma 2.2, then*

$$\int_0^{T_\star} \int_\Omega \Theta_{\varepsilon xxx}^2 \leq C, \quad \forall \varepsilon \in (0, \varepsilon_\star). \tag{3.1}$$

Proof. Testing the Eq. (2.7d) in a straightforward manner, we see that thanks to Young’s inequality,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_\Omega \Theta_{\varepsilon xx}^2 &= - \int_\Omega \Theta_{\varepsilon xxx} \Theta_{\varepsilon xt} \\ &= - \int_\Omega \Theta_{\varepsilon xxx} \cdot \{ D\Theta_{\varepsilon xxx} + 2\Gamma_\varepsilon(\Theta_\varepsilon) v_{\varepsilon x} v_{\varepsilon xx} + \Gamma'_\varepsilon(\Theta_\varepsilon) \Theta_{\varepsilon x} v_{\varepsilon x}^2 \} \\ &\leq - \frac{D}{2} \int_\Omega \Theta_{\varepsilon xxx}^2 + \frac{4}{D} \int_\Omega \Gamma_\varepsilon^2(\Theta_\varepsilon) v_{\varepsilon x}^2 v_{\varepsilon xx}^2 + \frac{1}{D} \int_\Omega \Gamma_\varepsilon'^2(\Theta_\varepsilon) \Theta_{\varepsilon x}^2 v_{\varepsilon x}^4 \end{aligned} \tag{3.2}$$

for all $t \in (0, T_{\max, \varepsilon})$ and $\varepsilon \in (0, 1)$. Here, (2.14) and (2.15) together with the fact that $\Gamma_\varepsilon \rightarrow \Gamma$ in $C^1([0, \infty))$ as $\varepsilon \searrow 0$ ensure the existence of $c_1 > 0$ and $c_2 > 0$ such that

$$\Gamma_\varepsilon^2(\Theta_\varepsilon) \leq c_1, \quad \Gamma_\varepsilon'^2(\Theta_\varepsilon)\Theta_{\varepsilon x}^2 \leq c_2 \quad \text{in } \Omega \times (0, T_\star), \quad \forall \varepsilon \in (0, \varepsilon_\star),$$

whereas (2.12) in conjunction with the continuity of the embedding $W^{2,2}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ provides $c_3 > 0$ and $c_4 > 0$ such that whenever $\varepsilon \in (0, \varepsilon_\star)$,

$$\int_\Omega v_{\varepsilon xx}^2 \leq c_3, \quad \forall t \in (0, T_\star), \quad |v_{\varepsilon x}| \leq c_4 \quad \text{in } \Omega \times (0, T_\star).$$

Consequently, (3.2) implies that

$$\frac{d}{dt} \int_\Omega \Theta_{\varepsilon xx}^2 + D \int_\Omega \Theta_{\varepsilon xxx}^2 \leq \frac{8c_1c_3c_4^2}{D} + \frac{2c_2c_4^4|\Omega|}{D}, \quad \forall t \in (0, T_\star), \quad \varepsilon \in (0, \varepsilon_\star),$$

which upon an integration using (2.5) entails (3.1), because $[0, T_{\max, \varepsilon}] \ni t \mapsto \int_\Omega \Theta_{\varepsilon xx}^2$ is continuous according to (2.8). □

A second preparation for our energy analysis utilizes parabolic theory in recording two regularity properties of third-order spatial derivatives. For fixed $\varepsilon \in (0, 1)$, in Lemma 3.3 these will be used in asserting continuity of an approximate counterpart of the functional in (1.6) at the initial instant.

Lemma 3.2. *For each $\varepsilon \in (0, 1)$ the solution of (2.7) from Lemma 2.1 has the additional properties that*

$$v_{\varepsilon xxx} \in C^0(\overline{\Omega} \times [0, T_{\max, \varepsilon}]), \tag{3.3}$$

$$u_{\varepsilon xxx} \in C^0(\overline{\Omega} \times [0, T_{\max, \varepsilon}]). \tag{3.4}$$

Proof. We take an arbitrary $\alpha \in (3/2, 2)$, and then choose $p > 1$ large enough and $\vartheta > 0$ suitably small such that

$$2\alpha - \frac{1}{p} > 3 + \vartheta, \tag{3.5}$$

and defining A as the realization of $-\varepsilon(\cdot)_{xx} + 1$ under homogeneous Neumann boundary conditions in $L^p(\Omega)$, with domain thus given by $D(A) = \{\varphi \in W^{2,p}(\Omega) \mid \varphi_x = 0 \text{ on } \partial\Omega\}$, we let A^α denote the corresponding fractional power. For fixed $\varepsilon \in (0, 1)$ and $T \in (0, T_{\max, \varepsilon})$, Lemma 2.1 then asserts that $Av_\varepsilon + Aw_\varepsilon$ is continuous in $\overline{\Omega} \times [0, T]$ and hence particularly belongs to $L^\infty((0, T); L^p(\Omega))$. As furthermore the inclusion $v_{0\varepsilon x} \in C_0^\infty(\Omega)$ warrants that $A^\alpha v_{0\varepsilon} \in L^p(\Omega)$, we can draw on standard

smoothing properties of the analytic semigroup $(e^{-tA})_{t \geq 0}$ [23] and the fact that for each $\beta \in \mathbb{R}$ the corresponding fractional power A^β commutes with e^{-tA} on $D(A^\beta)$. On the basis of a Duhamel representation associated with the identity $v_{\varepsilon t} = -Av_\varepsilon + v_\varepsilon + w_\varepsilon$, namely, we thereby see that with some $c_1(\varepsilon) > 0$ and $c_2(\varepsilon, T) > 0$ we have

$$\begin{aligned} \|A^\alpha v_\varepsilon(\cdot, t)\|_{L^p(\Omega)} &= \left\| A^\alpha e^{-tA} v_{0\varepsilon} + \int_0^t A^\alpha e^{-(t-s)A} \{v_\varepsilon(\cdot, s) + w_\varepsilon(\cdot, s)\} ds \right\|_{L^p(\Omega)} \\ &= \left\| e^{-tA} A^\alpha v_{0\varepsilon} + \int_0^t A^{\alpha-1} e^{-(t-s)A} A \{v_\varepsilon(\cdot, s) + w_\varepsilon(\cdot, s)\} ds \right\|_{L^p(\Omega)} \\ &\leq c_1(\varepsilon) \|A^\alpha v_{0\varepsilon}\|_{L^p(\Omega)} \\ &\quad + c_1(\varepsilon) \int_0^t (t-s)^{1-\alpha} \|Av_\varepsilon(\cdot, s) + Aw_\varepsilon(\cdot, s)\|_{L^p(\Omega)} ds \\ &\leq c_2(\varepsilon, T) + c_2(\varepsilon, T) \int_0^t (t-s)^{1-\alpha} ds, \quad \forall t \in (0, T). \end{aligned}$$

As $1-\alpha > -1$, we thus infer that $A^\alpha v_\varepsilon \in L^\infty((0, T); L^p(\Omega))$, whence using that (3.5) implies that $D(A^\alpha) \hookrightarrow C^{3+\vartheta}(\overline{\Omega})$ [25], we obtain that $v_\varepsilon \in L^\infty((0, T); C^{3+\vartheta}(\overline{\Omega}))$. As from Lemma 2.1 we already know that $v_\varepsilon \in C^0([0, T]; C^0(\overline{\Omega}))$, by compactness of the first of the two continuous embeddings $C^{3+\vartheta}(\overline{\Omega}) \hookrightarrow C^3(\overline{\Omega}) \hookrightarrow C^0(\Omega)$ we may employ a straightforward interpolation argument relying on an associated Ehrling type inequality to see that, in fact, $v_\varepsilon \in C^0([0, T]; C^3(\overline{\Omega}))$. Since $T \in (0, T_{\max, \varepsilon})$ was arbitrary, this entails (3.3), whereas (3.4) can be verified in much the same manner. \square

We are now prepared for the core step of our analysis, further developing the basic information from Lemma 2.2 so as to yield ε -independent estimates at an enhanced level of regularity.

Lemma 3.3. *Let ε_\star and T_\star be as in Lemma 2.2. Then there exist $\varepsilon_{\star\star} \in (0, \varepsilon_\star)$ and $C > 0$ such that*

$$\int_\Omega w_{\varepsilon xx}^2(\cdot, t) \leq C, \quad \forall t \in (0, T_\star), \quad \varepsilon \in (0, \varepsilon_{\star\star}), \quad (3.6)$$

$$\int_\Omega v_{\varepsilon xxx}^2(\cdot, t) \leq C, \quad \forall t \in (0, T_\star), \quad \varepsilon \in (0, \varepsilon_{\star\star}), \quad (3.7)$$

$$\int_\Omega u_{\varepsilon xxx}^2(\cdot, t) \leq C, \quad \forall t \in (0, T_\star), \quad \varepsilon \in (0, \varepsilon_{\star\star}). \quad (3.8)$$

Proof. Let us first employ Lemma 2.2 to fix positive constants $c_i, i \in \{1, \dots, 8\}$ such that

$$\int_{\Omega} v_{\varepsilon xx}^2 \leq c_1, \quad \int_{\Omega} u_{\varepsilon xx}^2 \leq c_2, \quad \forall t \in (0, T_\star), \quad \varepsilon \in (0, \varepsilon_\star), \quad (3.9)$$

$$|v_{\varepsilon x}| \leq c_3, \quad |u_{\varepsilon x}| \leq c_4 \quad \text{in } \Omega \times (0, T_\star), \quad \forall \varepsilon \in (0, \varepsilon_\star), \quad (3.10)$$

and that

$$\Theta_\varepsilon \leq c_5, \quad |\Theta_{\varepsilon x}| \leq c_6, \quad |\Theta_{\varepsilon xx}| \leq c_7, \quad |\Theta_{\varepsilon t}| \leq c_8 \quad \text{in } \Omega \times (0, T_\star), \quad \forall \varepsilon \in (0, \varepsilon_\star), \quad (3.11)$$

whence according to (2.4) we can find $\varepsilon_\bullet \in (0, \varepsilon_\star)$ and $c_i > 0, i \in \{9, \dots, 18\}$ such that

$$c_9 \leq \gamma_\varepsilon(\Theta_\varepsilon) \leq c_{10}, \quad c_{11} \leq \widehat{\gamma}_\varepsilon(\Theta_\varepsilon) \leq c_{12} \quad \text{in } \Omega \times (0, T_\star) \quad \forall \varepsilon \in (0, \varepsilon_\bullet), \quad (3.12)$$

that

$$|\gamma'_\varepsilon(\Theta_\varepsilon)| \leq c_{13}, \quad |\gamma''_\varepsilon(\Theta_\varepsilon)| \leq c_{14}, \quad |\gamma'''_\varepsilon(\Theta_\varepsilon)| \leq c_{15} \quad (3.13)$$

$$\text{in } \Omega \times (0, T_\star), \quad \forall \varepsilon \in (0, \varepsilon_\bullet),$$

$$|\widehat{\gamma}'_\varepsilon(\Theta_\varepsilon)| \leq c_{16}, \quad |\widehat{\gamma}''_\varepsilon(\Theta_\varepsilon)| \leq c_{17}, \quad |\widehat{\gamma}'''_\varepsilon(\Theta_\varepsilon)| \leq c_{18} \quad (3.14)$$

$$\text{in } \Omega \times (0, T_\star), \quad \forall \varepsilon \in (0, \varepsilon_\bullet).$$

We now go back to the Eq. (2.7a) to see upon multiplying by $w_{\varepsilon xxx}$ and integrating by parts that since $w_{\varepsilon x} = w_{\varepsilon xt} = 0$ on $\partial\Omega \times (0, T_{\max, \varepsilon})$ for all $\varepsilon \in (0, 1)$,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} w_{\varepsilon xx}^2 &= - \int_{\Omega} w_{\varepsilon xxx} w_{\varepsilon xt} \\ &= - \int_{\Omega} w_{\varepsilon xxx} \cdot \{ \varepsilon w_{\varepsilon xx} - \alpha w_\varepsilon + \gamma_\varepsilon(\Theta_\varepsilon) v_{\varepsilon xx} + \gamma'_\varepsilon(\Theta_\varepsilon) \Theta_{\varepsilon x} v_{\varepsilon x} \\ &\quad + \widehat{\gamma}_\varepsilon(\Theta_\varepsilon) u_{\varepsilon xx} + \widehat{\gamma}'_\varepsilon(\Theta_\varepsilon) \Theta_{\varepsilon x} u_{\varepsilon x} \} x \\ &= - \varepsilon \int_{\Omega} w_{\varepsilon xxx}^2 - \alpha \int_{\Omega} w_{\varepsilon xx}^2 \\ &\quad - \int_{\Omega} \gamma_\varepsilon(\Theta_\varepsilon) v_{\varepsilon xxx} w_{\varepsilon xxx} - 2 \int_{\Omega} \gamma'_\varepsilon(\Theta_\varepsilon) \Theta_{\varepsilon x} v_{\varepsilon xx} w_{\varepsilon xxx} \\ &\quad - \int_{\Omega} \gamma'_\varepsilon(\Theta_\varepsilon) \Theta_{\varepsilon xx} v_{\varepsilon x} w_{\varepsilon xxx} - \int_{\Omega} \gamma''_\varepsilon(\Theta_\varepsilon) \Theta_{\varepsilon x}^2 v_{\varepsilon x} w_{\varepsilon xxx} \\ &\quad - \int_{\Omega} \widehat{\gamma}_\varepsilon(\Theta_\varepsilon) u_{\varepsilon xxx} w_{\varepsilon xxx} - 2 \int_{\Omega} \widehat{\gamma}'_\varepsilon(\Theta_\varepsilon) \Theta_{\varepsilon x} u_{\varepsilon xx} w_{\varepsilon xxx} \\ &\quad - \int_{\Omega} \widehat{\gamma}'_\varepsilon(\Theta_\varepsilon) \Theta_{\varepsilon xx} u_{\varepsilon x} w_{\varepsilon xxx} - \int_{\Omega} \widehat{\gamma}''_\varepsilon(\Theta_\varepsilon) \Theta_{\varepsilon x}^2 u_{\varepsilon x} w_{\varepsilon xxx} \end{aligned} \quad (3.15)$$

for all $t \in (0, T_{\max, \varepsilon})$ and $\varepsilon \in (0, 1)$, and that here, in line with the Eq. (2.7b), Young's inequality as well as (3.11)-(3.13),

$$\begin{aligned}
& - \int_{\Omega} \gamma_{\varepsilon}(\Theta_{\varepsilon}) v_{\varepsilon xxx} w_{\varepsilon xxx} \\
&= - \int_{\Omega} \gamma_{\varepsilon}(\Theta_{\varepsilon}) v_{\varepsilon xxx} \cdot \{v_{\varepsilon xxx t} - \varepsilon v_{\varepsilon xxx x}\} \\
&= - \frac{1}{2} \frac{d}{dt} \int_{\Omega} \gamma_{\varepsilon}(\Theta_{\varepsilon}) v_{\varepsilon xxx}^2 + \frac{1}{2} \int_{\Omega} \gamma'_{\varepsilon}(\Theta_{\varepsilon}) \Theta_{\varepsilon t} v_{\varepsilon xxx}^2 \\
&\quad - \varepsilon \int_{\Omega} \gamma_{\varepsilon}(\Theta_{\varepsilon}) v_{\varepsilon xxx x}^2 - \varepsilon \int_{\Omega} \gamma'_{\varepsilon}(\Theta_{\varepsilon}) \Theta_{\varepsilon x} v_{\varepsilon xxx} v_{\varepsilon xxx x} \\
&\leq - \frac{1}{2} \frac{d}{dt} \int_{\Omega} \gamma_{\varepsilon}(\Theta_{\varepsilon}) v_{\varepsilon xxx}^2 + \frac{1}{2} \int_{\Omega} \gamma'_{\varepsilon}(\Theta_{\varepsilon}) \Theta_{\varepsilon t} v_{\varepsilon xxx}^2 \\
&\quad - \frac{\varepsilon}{2} \int_{\Omega} \gamma_{\varepsilon}(\Theta_{\varepsilon}) v_{\varepsilon xxx x}^2 + \frac{\varepsilon}{2} \int_{\Omega} \frac{\gamma_{\varepsilon}^{\prime 2}(\Theta_{\varepsilon})}{\gamma_{\varepsilon}(\Theta_{\varepsilon})} \Theta_{\varepsilon x}^2 v_{\varepsilon xxx}^2 \\
&\leq - \frac{1}{2} \frac{d}{dt} \int_{\Omega} \gamma_{\varepsilon}(\Theta_{\varepsilon}) v_{\varepsilon xxx}^2 - \frac{c_9 \varepsilon}{2} \int_{\Omega} v_{\varepsilon xxx x}^2 + c_{19} \int_{\Omega} \gamma_{\varepsilon}(\Theta_{\varepsilon}) v_{\varepsilon xxx}^2 \quad (3.16)
\end{aligned}$$

for all $t \in (0, T_{\star})$ and $\varepsilon \in (0, \varepsilon_{\bullet})$ with

$$c_{19} := \frac{c_8 c_{13}}{2c_9} + \frac{c_6^2 c_{13}^2}{2c_9^2},$$

because $\varepsilon_{\bullet} \leq 1$ and $v_{\varepsilon xxx} = 0$ on $\partial\Omega \times (0, T_{\max, \varepsilon})$ for all $\varepsilon \in (0, 1)$ by (2.7). Similarly, using that also $u_{\varepsilon xxx} = 0$ on $\partial\Omega \times (0, T_{\max, \varepsilon})$ for all $\varepsilon \in (0, 1)$, we find that for all $t \in (0, T_{\max, \varepsilon})$ and $\varepsilon \in (0, 1)$,

$$\begin{aligned}
& - \int_{\Omega} \widehat{\gamma}_{\varepsilon}(\Theta_{\varepsilon}) u_{\varepsilon xxx} w_{\varepsilon xxx} \\
&= - \int_{\Omega} \widehat{\gamma}_{\varepsilon}(\Theta_{\varepsilon}) u_{\varepsilon xxx} \cdot \{v_{\varepsilon xxx t} - \varepsilon v_{\varepsilon xxx x}\} \\
&= - \frac{d}{dt} \int_{\Omega} \widehat{\gamma}_{\varepsilon}(\Theta_{\varepsilon}) u_{\varepsilon xxx} v_{\varepsilon xxx} + \int_{\Omega} \widehat{\gamma}_{\varepsilon}(\Theta_{\varepsilon}) u_{\varepsilon xxx t} v_{\varepsilon xxx} \\
&\quad + \int_{\Omega} \widehat{\gamma}'_{\varepsilon}(\Theta_{\varepsilon}) \Theta_{\varepsilon t} u_{\varepsilon xxx} v_{\varepsilon xxx} \\
&\quad - \varepsilon \int_{\Omega} \widehat{\gamma}_{\varepsilon}(\Theta_{\varepsilon}) u_{\varepsilon xxx x} v_{\varepsilon xxx x} - \varepsilon \int_{\Omega} \widehat{\gamma}'_{\varepsilon}(\Theta_{\varepsilon}) \Theta_{\varepsilon x} u_{\varepsilon xxx} v_{\varepsilon xxx x},
\end{aligned}$$

where by the Eq. (2.7c) and, again, the fact that $v_{\varepsilon xxx} = 0$ on $\partial\Omega \times (0, T_{\max, \varepsilon})$ for all $\varepsilon \in (0, 1)$,

$$\int_{\Omega} \widehat{\gamma}_{\varepsilon}(\Theta_{\varepsilon}) u_{\varepsilon xxx t} v_{\varepsilon xxx}$$

$$\begin{aligned}
 &= \int_{\Omega} \widehat{\gamma}_{\varepsilon}(\Theta_{\varepsilon}) \cdot \{v_{\varepsilon xxx} + \varepsilon u_{\varepsilon xxxxx}\} \cdot v_{\varepsilon xxx} \\
 &= \int_{\Omega} \widehat{\gamma}_{\varepsilon}(\Theta_{\varepsilon}) v_{\varepsilon xxx}^2 - \varepsilon \int_{\Omega} \widehat{\gamma}_{\varepsilon}(\Theta_{\varepsilon}) u_{\varepsilon xxx} v_{\varepsilon xxx} \\
 &\quad - \varepsilon \int_{\Omega} \widehat{\gamma}'_{\varepsilon}(\Theta_{\varepsilon}) \Theta_{\varepsilon x} u_{\varepsilon xxx} v_{\varepsilon xxx}
 \end{aligned}$$

for all $t \in (0, T_{\max, \varepsilon})$ and $\varepsilon \in (0, 1)$. Once more due to Young's inequality and the restriction that $\varepsilon_{\bullet} \leq 1$, and thanks to (3.12), (3.14) and (3.11), we thereby see that

$$\begin{aligned}
 & - \int_{\Omega} \widehat{\gamma}_{\varepsilon}(\Theta_{\varepsilon}) u_{\varepsilon xxx} w_{\varepsilon xxx} \\
 &= - \frac{d}{dt} \int_{\Omega} \widehat{\gamma}_{\varepsilon}(\Theta_{\varepsilon}) u_{\varepsilon xxx} v_{\varepsilon xxx} + \int_{\Omega} \widehat{\gamma}_{\varepsilon}(\Theta_{\varepsilon}) v_{\varepsilon xxx}^2 \\
 &\quad + \int_{\Omega} \widehat{\gamma}'_{\varepsilon}(\Theta_{\varepsilon}) \Theta_{\varepsilon t} u_{\varepsilon xxx} v_{\varepsilon xxx} - 2\varepsilon \int_{\Omega} \widehat{\gamma}_{\varepsilon}(\Theta_{\varepsilon}) u_{\varepsilon xxx} v_{\varepsilon xxx} \\
 &\quad - \varepsilon \int_{\Omega} \widehat{\gamma}'_{\varepsilon}(\Theta_{\varepsilon}) \Theta_{\varepsilon x} u_{\varepsilon xxx} v_{\varepsilon xxx} - \varepsilon \int_{\Omega} \widehat{\gamma}'_{\varepsilon}(\Theta_{\varepsilon}) \Theta_{\varepsilon x} u_{\varepsilon xxx} v_{\varepsilon xxx} \\
 &\leq - \frac{d}{dt} \int_{\Omega} \widehat{\gamma}_{\varepsilon}(\Theta_{\varepsilon}) u_{\varepsilon xxx} v_{\varepsilon xxx} + \int_{\Omega} \widehat{\gamma}_{\varepsilon}(\Theta_{\varepsilon}) v_{\varepsilon xxx}^2 + \int_{\Omega} \gamma_{\varepsilon}(\Theta_{\varepsilon}) v_{\varepsilon xxx}^2 \\
 &\quad + \frac{1}{4} \int_{\Omega} \frac{\widehat{\gamma}_{\varepsilon}^{\prime 2}(\Theta_{\varepsilon})}{\gamma_{\varepsilon}(\Theta_{\varepsilon})} \Theta_{\varepsilon t}^2 u_{\varepsilon xxx}^2 + \frac{c_9 \varepsilon}{4} \int_{\Omega} v_{\varepsilon xxx}^2 + \frac{4\varepsilon}{c_9} \int_{\Omega} \widehat{\gamma}_{\varepsilon}^2(\Theta_{\varepsilon}) u_{\varepsilon xxx}^2 \\
 &\quad + \frac{c_9 \varepsilon}{4} \int_{\Omega} v_{\varepsilon xxx}^2 + \frac{\varepsilon}{c_9} \int_{\Omega} \widehat{\gamma}_{\varepsilon}^{\prime 2}(\Theta_{\varepsilon}) \Theta_{\varepsilon x}^2 u_{\varepsilon xxx}^2 \\
 &\quad + \int_{\Omega} \gamma_{\varepsilon}(\Theta_{\varepsilon}) v_{\varepsilon xxx}^2 + \frac{\varepsilon^2}{4} \int_{\Omega} \frac{\widehat{\gamma}_{\varepsilon}^{\prime 2}(\Theta_{\varepsilon})}{\gamma_{\varepsilon}(\Theta_{\varepsilon})} \Theta_{\varepsilon x}^2 u_{\varepsilon xxx}^2 \\
 &\leq - \frac{d}{dt} \int_{\Omega} \widehat{\gamma}_{\varepsilon}(\Theta_{\varepsilon}) u_{\varepsilon xxx} v_{\varepsilon xxx} + c_{20} \int_{\Omega} \gamma_{\varepsilon}(\Theta_{\varepsilon}) v_{\varepsilon xxx}^2 + c_{21} \int_{\Omega} u_{\varepsilon xxx}^2 \\
 &\quad + c_{22} \varepsilon \int_{\Omega} u_{\varepsilon xxx}^2 + \frac{c_9 \varepsilon}{2} \int_{\Omega} v_{\varepsilon xxx}^2, \quad \forall t \in (0, T_{\star}), \quad \varepsilon \in (0, \varepsilon_{\bullet}) \tag{3.17}
 \end{aligned}$$

with

$$c_{20} := \frac{c_{12}}{c_9} + 2, \quad c_{21} := \frac{c_8^2 c_{16}^2}{4c_9} + \frac{c_6^2 c_{16}^2}{c_9}, \quad c_{22} := \frac{4c_{12}^2}{c_9} + \frac{c_6^2 c_{16}^2}{4c_9}.$$

In the fourth, fifth and sixth summands on the right of (3.15), we first integrate by parts and then again employ Young's inequality to see, repeatedly using (3.9)-(3.13), that for all $t \in (0, T_{\star})$ and $\varepsilon \in (0, \varepsilon_{\bullet})$,

$$-2 \int_{\Omega} \widehat{\gamma}'_{\varepsilon}(\Theta_{\varepsilon}) \Theta_{\varepsilon x} v_{\varepsilon xx} w_{\varepsilon xxx}$$

$$\begin{aligned}
&= 2 \int_{\Omega} \gamma'_{\varepsilon}(\Theta_{\varepsilon}) \Theta_{\varepsilon x} v_{\varepsilon x x x} w_{\varepsilon x x} + 2 \int_{\Omega} \gamma'_{\varepsilon}(\Theta_{\varepsilon}) \Theta_{\varepsilon x x} v_{\varepsilon x x} w_{\varepsilon x x} \\
&\quad + 2 \int_{\Omega} \gamma''_{\varepsilon}(\Theta_{\varepsilon}) \Theta_{\varepsilon x}^2 v_{\varepsilon x x} w_{\varepsilon x x} \\
&\leq 3 \int_{\Omega} w_{\varepsilon x x}^2 + \int_{\Omega} \gamma_{\varepsilon}^{\prime 2}(\Theta_{\varepsilon}) \Theta_{\varepsilon x}^2 v_{\varepsilon x x}^2 \\
&\quad + \int_{\Omega} \gamma_{\varepsilon}^{\prime 2}(\Theta_{\varepsilon}) \Theta_{\varepsilon x x}^2 v_{\varepsilon x x}^2 + \int_{\Omega} \gamma_{\varepsilon}^{\prime \prime 2}(\Theta_{\varepsilon}) \Theta_{\varepsilon x}^4 v_{\varepsilon x x}^2 \\
&\leq 3 \int_{\Omega} w_{\varepsilon x x}^2 + c_{23} \int_{\Omega} \gamma_{\varepsilon}(\Theta_{\varepsilon}) v_{\varepsilon x x x}^2 + c_{24} \tag{3.18}
\end{aligned}$$

with

$$c_{23} := \frac{c_6^2 c_{13}^2}{c_9}, \quad c_{24} := c_1 c_7^2 c_{13}^2 + c_1 c_6^2 c_{14}^2,$$

that

$$\begin{aligned}
&\quad - \int_{\Omega} \gamma'_{\varepsilon}(\Theta_{\varepsilon}) \Theta_{\varepsilon x x} v_{\varepsilon x} w_{\varepsilon x x x} \\
&= \int_{\Omega} \gamma'_{\varepsilon}(\Theta_{\varepsilon}) \Theta_{\varepsilon x x} v_{\varepsilon x x} w_{\varepsilon x x} + \int_{\Omega} \gamma'_{\varepsilon}(\Theta_{\varepsilon}) \Theta_{\varepsilon x x x} v_{\varepsilon x} w_{\varepsilon x x} \\
&\quad + \int_{\Omega} \gamma''_{\varepsilon}(\Theta_{\varepsilon}) \Theta_{\varepsilon x} \Theta_{\varepsilon x x} v_{\varepsilon x} w_{\varepsilon x x} \\
&\leq 3 \int_{\Omega} w_{\varepsilon x x}^2 + \frac{1}{4} \int_{\Omega} \gamma_{\varepsilon}^{\prime 2}(\Theta_{\varepsilon}) \Theta_{\varepsilon x x}^2 v_{\varepsilon x x}^2 \\
&\quad + \frac{1}{4} \int_{\Omega} \gamma_{\varepsilon}^{\prime 2}(\Theta_{\varepsilon}) \Theta_{\varepsilon x x x}^2 v_{\varepsilon x}^2 + \frac{1}{4} \int_{\Omega} \gamma_{\varepsilon}^{\prime \prime 2}(\Theta_{\varepsilon}) \Theta_{\varepsilon x}^2 \Theta_{\varepsilon x x}^2 v_{\varepsilon x}^2 \\
&\leq 3 \int_{\Omega} w_{\varepsilon x x}^2 + c_{25} \int_{\Omega} \Theta_{\varepsilon x x x}^2 + c_{26}, \quad \forall t \in (0, T_{\star}), \quad \varepsilon \in (0, \varepsilon_{\bullet}) \tag{3.19}
\end{aligned}$$

with

$$c_{25} := \frac{c_3^2 c_{13}^2}{4}, \quad c_{26} := \frac{c_1 c_7^2 c_{13}^2}{4} + \frac{c_3^2 c_6^2 c_7^2 c_{14}^2 |\Omega|}{4},$$

and that

$$\begin{aligned}
&\quad - \int_{\Omega} \gamma''_{\varepsilon}(\Theta_{\varepsilon}) \Theta_{\varepsilon x}^2 v_{\varepsilon x} w_{\varepsilon x x x} \\
&= \int_{\Omega} \gamma''_{\varepsilon}(\Theta_{\varepsilon}) \Theta_{\varepsilon x}^2 v_{\varepsilon x x} w_{\varepsilon x x} + 2 \int_{\Omega} \gamma''_{\varepsilon}(\Theta_{\varepsilon}) \Theta_{\varepsilon x} \Theta_{\varepsilon x x} v_{\varepsilon x} w_{\varepsilon x x} \\
&\quad + \int_{\Omega} \gamma_{\varepsilon}^{\prime \prime \prime}(\Theta_{\varepsilon}) \Theta_{\varepsilon x}^3 v_{\varepsilon x} w_{\varepsilon x x}
\end{aligned}$$

$$\begin{aligned}
 &\leq 3 \int_{\Omega} w_{\varepsilon xx}^2 + \frac{1}{4} \int_{\Omega} \gamma_{\varepsilon}''^2(\Theta_{\varepsilon}) \Theta_{\varepsilon x}^4 v_{\varepsilon xx}^2 \\
 &\quad + \int_{\Omega} \gamma_{\varepsilon}''^2(\Theta_{\varepsilon}) \Theta_{\varepsilon x}^2 \Theta_{\varepsilon xx}^2 v_{\varepsilon x}^2 + \frac{1}{4} \int_{\Omega} \gamma_{\varepsilon}''^2(\Theta_{\varepsilon}) \Theta_{\varepsilon x}^6 v_{\varepsilon x}^2 \\
 &\leq 3 \int_{\Omega} w_{\varepsilon xx}^2 + c_{27}, \quad \forall t \in (0, T_{*}), \quad \varepsilon \in (0, \varepsilon_{\bullet})
 \end{aligned} \tag{3.20}$$

with

$$c_{27} := \frac{c_1 c_6^4 c_{14}^2}{4} + c_3^2 c_6^2 c_7^2 c_{14}^2 |\Omega| + \frac{c_3^2 c_6^6 c_{15}^2 |\Omega|}{4}.$$

In quite the same fashion, we can estimate the three rightmost expressions in (3.15) so as to verify that

$$\begin{aligned}
 &-2 \int_{\Omega} \widehat{\gamma}_{\varepsilon}'(\Theta_{\varepsilon}) \Theta_{\varepsilon x} u_{\varepsilon xx} w_{\varepsilon xxx} \\
 &= 2 \int_{\Omega} \widehat{\gamma}_{\varepsilon}'(\Theta_{\varepsilon}) \Theta_{\varepsilon x} u_{\varepsilon xxx} w_{\varepsilon xx} + 2 \int_{\Omega} \widehat{\gamma}_{\varepsilon}'(\Theta_{\varepsilon}) \Theta_{\varepsilon xx} u_{\varepsilon xx} w_{\varepsilon xx} \\
 &\quad + 2 \int_{\Omega} \widehat{\gamma}_{\varepsilon}''(\Theta_{\varepsilon}) \Theta_{\varepsilon x}^2 u_{\varepsilon xx} w_{\varepsilon xx} \\
 &\leq 3 \int_{\Omega} w_{\varepsilon xx}^2 + \int_{\Omega} \widehat{\gamma}_{\varepsilon}''^2(\Theta_{\varepsilon}) \Theta_{\varepsilon x}^2 u_{\varepsilon xxx}^2 \\
 &\quad + \int_{\Omega} \widehat{\gamma}_{\varepsilon}''^2(\Theta_{\varepsilon}) \Theta_{\varepsilon xx}^2 u_{\varepsilon xx}^2 + \int_{\Omega} \widehat{\gamma}_{\varepsilon}''^2(\Theta_{\varepsilon}) \Theta_{\varepsilon x}^4 u_{\varepsilon xx}^2 \\
 &\leq 3 \int_{\Omega} w_{\varepsilon xx}^2 + c_{28} \int_{\Omega} u_{\varepsilon xxx}^2 + c_{29}, \quad \forall t \in (0, T_{*}), \quad \varepsilon \in (0, \varepsilon_{\bullet})
 \end{aligned} \tag{3.21}$$

with

$$c_{28} := c_6^2 c_{16}^2, \quad c_{29} := c_2 c_7^2 c_{16}^2 + c_2 c_6^4 c_{17}^2,$$

that

$$\begin{aligned}
 &- \int_{\Omega} \widehat{\gamma}_{\varepsilon}'(\Theta_{\varepsilon}) \Theta_{\varepsilon xx} u_{\varepsilon x} w_{\varepsilon xxx} \\
 &= \int_{\Omega} \widehat{\gamma}_{\varepsilon}'(\Theta_{\varepsilon}) \Theta_{\varepsilon xx} u_{\varepsilon xx} w_{\varepsilon xx} + \int_{\Omega} \widehat{\gamma}_{\varepsilon}'(\Theta_{\varepsilon}) \Theta_{\varepsilon xxx} u_{\varepsilon x} w_{\varepsilon xx} \\
 &\quad + \int_{\Omega} \widehat{\gamma}_{\varepsilon}''(\Theta_{\varepsilon}) \Theta_{\varepsilon x} \Theta_{\varepsilon xx} u_{\varepsilon x} w_{\varepsilon xx} \\
 &\leq 3 \int_{\Omega} w_{\varepsilon xx}^2 + \frac{1}{4} \int_{\Omega} \widehat{\gamma}_{\varepsilon}''^2(\Theta_{\varepsilon}) \Theta_{\varepsilon xx}^2 u_{\varepsilon xx}^2 \\
 &\quad + \frac{1}{4} \int_{\Omega} \widehat{\gamma}_{\varepsilon}''^2(\Theta_{\varepsilon}) \Theta_{\varepsilon xxx}^2 u_{\varepsilon x}^2 + \frac{1}{4} \int_{\Omega} \widehat{\gamma}_{\varepsilon}''^2(\Theta_{\varepsilon}) \Theta_{\varepsilon x}^2 \Theta_{\varepsilon xx}^2 u_{\varepsilon x}^2
 \end{aligned}$$

$$\leq 3 \int_{\Omega} w_{\varepsilon x x}^2 + c_{30} \int_{\Omega} \Theta_{\varepsilon x x x}^2 + c_{31}, \quad \forall t \in (0, T_{\star}), \quad \varepsilon \in (0, \varepsilon_{\bullet}) \tag{3.22}$$

with

$$c_{30} := \frac{c_4^2 c_{16}^2}{4}, \quad c_{31} := \frac{c_2 c_7^2 c_{16}^2}{4} + \frac{c_4^2 c_6^2 c_7^2 c_{17}^2 |\Omega|}{4},$$

and that for all $t \in (0, T_{\star})$ and $\varepsilon \in (0, \varepsilon_{\bullet})$,

$$\begin{aligned} & - \int_{\Omega} \widehat{\gamma}_{\varepsilon}''(\Theta_{\varepsilon}) \Theta_{\varepsilon x}^2 u_{\varepsilon x} w_{\varepsilon x x x} \\ &= \int_{\Omega} \widehat{\gamma}_{\varepsilon}''(\Theta_{\varepsilon}) \Theta_{\varepsilon x}^2 u_{\varepsilon x x} w_{\varepsilon x x} + 2 \int_{\Omega} \widehat{\gamma}_{\varepsilon}''(\Theta_{\varepsilon}) \Theta_{\varepsilon x} \Theta_{\varepsilon x x} u_{\varepsilon x} w_{\varepsilon x x} \\ & \quad + \int_{\Omega} \widehat{\gamma}_{\varepsilon}'''(\Theta_{\varepsilon}) \Theta_{\varepsilon x}^3 u_{\varepsilon x} w_{\varepsilon x x} \\ &\leq 3 \int_{\Omega} w_{\varepsilon x x}^2 + \frac{1}{4} \int_{\Omega} \widehat{\gamma}_{\varepsilon}''^2(\Theta_{\varepsilon}) \Theta_{\varepsilon x}^4 u_{\varepsilon x x}^2 \\ & \quad + \int_{\Omega} \widehat{\gamma}_{\varepsilon}''^2(\Theta_{\varepsilon}) \Theta_{\varepsilon x}^2 \Theta_{\varepsilon x x}^2 u_{\varepsilon x}^2 + \frac{1}{4} \int_{\Omega} \widehat{\gamma}_{\varepsilon}'''^2(\Theta_{\varepsilon}) \Theta_{\varepsilon x}^6 u_{\varepsilon x}^2 \\ &\leq 3 \int_{\Omega} w_{\varepsilon x x}^2 + c_{32}, \end{aligned} \tag{3.23}$$

where

$$c_{32} := \frac{c_2 c_6^4 c_{17}^2}{4} + c_4^2 c_6^2 c_7^2 c_{17}^2 |\Omega| + \frac{c_4^2 c_6^6 c_{18}^2 |\Omega|}{4}.$$

In summary, from (3.15)-(3.23) we obtain that for all $t \in (0, T_{\star})$ and $\varepsilon \in (0, \varepsilon_{\bullet})$,

$$\begin{aligned} & \frac{d}{dt} \left\{ \frac{1}{2} \int_{\Omega} w_{\varepsilon x x}^2 + \frac{1}{2} \int_{\Omega} \gamma_{\varepsilon}(\Theta_{\varepsilon}) v_{\varepsilon x x x}^2 + \int_{\Omega} \widehat{\gamma}_{\varepsilon}(\Theta_{\varepsilon}) u_{\varepsilon x x x} v_{\varepsilon x x x} \right\} \\ & \quad + \varepsilon \int_{\Omega} w_{\varepsilon x x x}^2 + \alpha \int_{\Omega} w_{\varepsilon x x}^2 \\ &\leq 18 \int_{\Omega} w_{\varepsilon x x}^2 + (c_{19} + c_{20} + c_{23}) \int_{\Omega} \gamma_{\varepsilon}(\Theta_{\varepsilon}) v_{\varepsilon x x x}^2 \\ & \quad + (c_{21} + c_{28}) \int_{\Omega} u_{\varepsilon x x x}^2 + c_{22} \varepsilon \int_{\Omega} u_{\varepsilon x x x x}^2 \\ & \quad + (c_{25} + c_{30}) \int_{\Omega} \Theta_{\varepsilon x x x}^2 + c_{24} + c_{26} + c_{27} + c_{29} + c_{31} + c_{32}, \end{aligned} \tag{3.24}$$

and in order to suitably cope with the third and fourth summand on the right-hand side herein, we once more use the Eq. (2.7c) to see in a straightforward manner that, again since $u_{\varepsilon x x x} = 0$ on $\partial\Omega \times (0, T_{\max, \varepsilon})$ for all $\varepsilon \in (0, 1)$,

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} u_{\varepsilon x x x}^2 = \int_{\Omega} u_{\varepsilon x x x} \cdot \{ \varepsilon u_{\varepsilon x x x x x} + v_{\varepsilon x x x} \}$$

$$\begin{aligned}
 &= -\varepsilon \int_{\Omega} u_{\varepsilon x x x}^2 + \int_{\Omega} u_{\varepsilon x x x} v_{\varepsilon x x x} \\
 &\leq -\varepsilon \int_{\Omega} u_{\varepsilon x x x}^2 + \frac{1}{2} \int_{\Omega} u_{\varepsilon x x x}^2 + \frac{1}{2} \int_{\Omega} v_{\varepsilon x x x}^2 \\
 &\leq -\varepsilon \int_{\Omega} u_{\varepsilon x x x}^2 + \frac{1}{2} \int_{\Omega} u_{\varepsilon x x x}^2 + \frac{1}{2c_9} \int_{\Omega} \gamma_{\varepsilon}(\Theta_{\varepsilon}) v_{\varepsilon x x x}^2
 \end{aligned}$$

for all $t \in (0, T_{\star})$ and $\varepsilon \in (0, \varepsilon_{\bullet})$. Choosing $B > 0$ large enough such that

$$B \geq c_{22}, \quad \frac{B}{4} \geq \frac{c_{12}^2}{c_9}, \tag{3.25}$$

from (3.24), we thus infer upon dropping two nonnegative summands on its left-hand side and using the first inequality in (3.25) that

$$\begin{aligned}
 y_{\varepsilon}(t) &:= \frac{1}{2} \int_{\Omega} w_{\varepsilon x x}^2 + \frac{1}{2} \int_{\Omega} \gamma_{\varepsilon}(\Theta_{\varepsilon}) v_{\varepsilon x x x}^2 + \int_{\Omega} \widehat{\gamma}_{\varepsilon}(\Theta_{\varepsilon}) u_{\varepsilon x x x} v_{\varepsilon x x x} \\
 &\quad + \frac{B}{2} \int_{\Omega} u_{\varepsilon x x x}^2, \quad t \in [0, T_{\star}), \quad \varepsilon \in (0, \varepsilon_{\bullet}),
 \end{aligned}$$

satisfies

$$y'_{\varepsilon}(t) \leq 18 \int_{\Omega} w_{\varepsilon x x}^2 + c_{33} \int_{\Omega} \gamma_{\varepsilon}(\Theta_{\varepsilon}) v_{\varepsilon x x x}^2 + c_{34} \int_{\Omega} u_{\varepsilon x x x}^2 + h_{\varepsilon}(t) \tag{3.26}$$

for all $t \in (0, T_{\star})$ and $\varepsilon \in (0, \varepsilon_{\bullet})$, where

$$c_{33} := c_{19} + c_{20} + c_{23} + \frac{B}{2c_9}, \quad c_{34} := c_{21} + c_{28} + \frac{B}{2},$$

and where Lemma 3.1 yields $c_{35} > 0$ such that

$$h_{\varepsilon}(t) := (c_{25} + c_{30}) \int_{\Omega} \Theta_{\varepsilon x x x}^2 + c_{24} + c_{26} + c_{27} + c_{29} + c_{31} + c_{32}, \quad t \in (0, T_{\star}), \quad \varepsilon \in (0, \varepsilon_{\bullet})$$

has the property that

$$\int_0^{T_{\star}} h_{\varepsilon}(t) dt \leq c_{35}, \quad \forall \varepsilon \in (0, \varepsilon_{\bullet}). \tag{3.27}$$

Now thanks to (3.12) and the second restriction in (3.25), Young’s inequality en-

sure that for all $t \in (0, T_*)$ and $\varepsilon \in (0, \varepsilon_\bullet)$,

$$\begin{aligned} y_\varepsilon(t) &\geq \frac{1}{2} \int_{\Omega} w_{\varepsilon xx}^2 + \frac{1}{4} \int_{\Omega} \gamma_\varepsilon(\Theta_\varepsilon) v_{\varepsilon xxx}^2 - \int_{\Omega} \frac{\widehat{\gamma}_\varepsilon^2(\Theta_\varepsilon)}{\gamma_\varepsilon(\Theta_\varepsilon)} u_{\varepsilon xxx}^2 + \frac{B}{2} \int_{\Omega} u_{\varepsilon xxx}^2 \\ &\geq \frac{1}{2} \int_{\Omega} w_{\varepsilon xx}^2 + \frac{1}{4} \int_{\Omega} \gamma_\varepsilon(\Theta_\varepsilon) v_{\varepsilon xxx}^2 - \frac{c_{12}^2}{c_9} \int_{\Omega} u_{\varepsilon xxx}^2 + \frac{B}{2} \int_{\Omega} u_{\varepsilon xxx}^2 \\ &\geq \frac{1}{2} \int_{\Omega} w_{\varepsilon xx}^2 + \frac{1}{4} \int_{\Omega} \gamma_\varepsilon(\Theta_\varepsilon) v_{\varepsilon xxx}^2 + \frac{B}{4} \int_{\Omega} u_{\varepsilon xxx}^2, \end{aligned} \quad (3.28)$$

so that from (3.26) we infer that if we let

$$c_{36} := 36 + 4c_{33} + \frac{4c_{34}}{B},$$

then

$$y'_\varepsilon(t) \leq c_{36} y_\varepsilon(t) + h_\varepsilon(t), \quad \forall t \in (0, T_*), \quad \varepsilon \in (0, \varepsilon_\bullet),$$

and hence,

$$\begin{aligned} y_\varepsilon(t) &\leq y_\varepsilon(0) \cdot e^{c_{36}t} + \int_0^t e^{c_{36}(t-s)} h_\varepsilon(s) ds \\ &\leq y_\varepsilon(0) \cdot e^{c_{36}T_*} + c_{35} e^{c_{36}T_*}, \quad \forall t \in (0, T_*), \quad \varepsilon \in (0, \varepsilon_\bullet) \end{aligned}$$

due to (3.27), because Lemma 3.2 in conjunction with (2.8) ensures continuity of y_ε on $[0, T_*]$. Since $\limsup_{\varepsilon \searrow 0} y_\varepsilon(0)$ is finite by (2.5) and (2.4), and since thus $\sup_{\varepsilon \in (0, \widehat{\varepsilon})} y_\varepsilon(0) < \infty$ for all suitably small $\widehat{\varepsilon} \in (0, 1)$, in view of (3.28) this readily entails the existence of $\varepsilon_{**} \in (0, \varepsilon_\bullet)$ and $C > 0$ such that (3.6)-(3.8) hold. \square

4 Proof of Theorem 1.1

Combining the estimates from Lemma 3.3 with suitable embedding properties available in the considered one-dimensional setting, still for initial data with vanishing averages we can readily confirm that under the current regularity assumptions on the data the strong solution approximated in Lemma 2.3 in fact exhibits smoothness properties compatible with those from the statement of Theorem 1.1.

Lemma 4.1. *If (2.1) holds, then the strong solution (u, Θ) obtained in Proposition 1.1 has the additional properties that*

$$u \in C^0([0, T_*]; C^2(\overline{\Omega})) \cap L^\infty((0, T_*); W^{3,2}(\Omega)), \quad (4.1)$$

$$u_t \in C^0([0, T_*]; C^2(\overline{\Omega})) \cap L^\infty((0, T_*); W^{3,2}(\Omega)), \quad (4.2)$$

$$u_{tt} \in C^0([0, T_*]; C^1(\overline{\Omega})) \cap L^\infty((0, T_*); W^{2,2}(\Omega)) \quad (4.3)$$

as well as

$$u_{ttt} \in C^0(\overline{\Omega} \times [0, T_*]) \cap L^\infty((0, T_*); W^{1,2}(\Omega)), \tag{4.4}$$

and (u, Θ) actually solves (1.1) in the classical sense in $\Omega \times (0, T_*)$.

Proof. From (3.8) and Lemma 2.2 we obtain $c_1 > 0$ such that with ε_{**} found in Lemma 3.3 we have

$$\|u_\varepsilon(\cdot, t)\|_{W^{3,2}(\Omega)} \leq c_1, \quad \forall t \in [0, T_*], \quad \varepsilon \in (0, \varepsilon_{**}), \tag{4.5}$$

and we claim that this implies that if we let $(\varepsilon_j)_{j \in \mathbb{N}} \subset (0, 1)$ be as in Lemma 2.3, then

$$u_\varepsilon \rightarrow u \quad \text{in } C^0([0, T_*]; C^2(\overline{\Omega})) \quad \text{as } \varepsilon = \varepsilon_j \searrow 0. \tag{4.6}$$

In fact, given $\eta > 0$ we may rely on the compactness of the embedding $W^{3,2}(\Omega) \hookrightarrow C^2(\overline{\Omega})$ to find $c_2(\eta) > 0$ such that

$$\|\varphi\|_{C^2(\overline{\Omega})} \leq \frac{\eta}{4c_1} \|\varphi\|_{W^{3,2}(\Omega)} + c_2(\eta) \|\varphi\|_{C^1(\overline{\Omega})}, \quad \forall \varphi \in W^{3,2}(\Omega), \tag{4.7}$$

and we thereupon make use of (2.16) in fixing $\varepsilon_\eta \in (0, 1)$ such that

$$c_2(\eta) \|u_\varepsilon(\cdot, t) - u_{\varepsilon'}(\cdot, t)\|_{C^1(\overline{\Omega})} \leq \frac{\eta}{2} \tag{4.8}$$

for all $t \in [0, T_*], \varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}} \cap (0, \varepsilon_\eta), \varepsilon' \in (\varepsilon_j)_{j \in \mathbb{N}} \cap (0, \varepsilon_\eta)$. A combination of (4.7) with (4.8) then shows that

$$\begin{aligned} & \|u_\varepsilon(\cdot, t) - u_{\varepsilon'}(\cdot, t)\|_{C^2(\overline{\Omega})} \\ & \leq \frac{\eta}{4c_1} \|u_\varepsilon(\cdot, t) - u_{\varepsilon'}(\cdot, t)\|_{W^{3,2}(\Omega)} + c_2(\eta) \|u_\varepsilon(\cdot, t) - u_{\varepsilon'}(\cdot, t)\|_{C^1(\overline{\Omega})} \\ & \leq \frac{\eta}{4c_1} \cdot \{ \|u_\varepsilon(\cdot, t)\|_{W^{3,2}(\Omega)} + \|u_{\varepsilon'}(\cdot, t)\|_{W^{3,2}(\Omega)} \} + c_2(\eta) \|u_\varepsilon(\cdot, t) - u_{\varepsilon'}(\cdot, t)\|_{C^1(\overline{\Omega})} \\ & \leq \frac{\eta}{4c_1} \cdot (c_1 + c_1) + \frac{\eta}{2}, \quad \forall t \in [0, T_*], \quad \forall \varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}} \cap (0, \varepsilon_\eta), \quad \varepsilon' \in (\varepsilon_j)_{j \in \mathbb{N}} \cap (0, \varepsilon_\eta) \end{aligned}$$

meaning that $(u_{\varepsilon_j})_{j \in \mathbb{N}}$ forms a Cauchy sequence in the Banach space $C^0([0, T_*]; C^2(\overline{\Omega}))$ which hence must satisfy (4.6).

Likewise, (3.7) in conjunction with Lemma 2.2 provides $c_3 > 0$ such that

$$\|v_\varepsilon(\cdot, t)\|_{W^{3,2}(\Omega)} \leq c_3, \quad \forall t \in [0, T], \quad \varepsilon \in (0, \varepsilon_{**}),$$

whence arguing as above we infer from the convergence property in (2.17) that

$$v_\varepsilon \rightarrow u_t \quad \text{in } C^0([0, T_*]; C^2(\overline{\Omega})) \quad \text{as } \varepsilon = \varepsilon_j \searrow 0. \quad (4.9)$$

Finally, (3.6) and Lemma 2.2 warrant the existence of $c_4 > 0$ fulfilling

$$\|w_\varepsilon(\cdot, t)\|_{W^{2,2}(\Omega)} \leq c_4, \quad \forall t \in [0, T], \quad \varepsilon \in (0, \varepsilon_{**}),$$

and now drawing on the compactness of the embedding $W^{2,2}(\Omega) \hookrightarrow C^1(\overline{\Omega})$ and the continuity of the inclusion $C^1(\overline{\Omega}) \hookrightarrow C^0(\overline{\Omega})$, we therefore obtain recalling (2.18) that

$$w_\varepsilon \rightarrow u_{tt} \quad \text{in } C^0([0, T_*]; C^1(\overline{\Omega})) \quad \text{as } \varepsilon = \varepsilon_j \searrow 0. \quad (4.10)$$

The inclusions in (4.1)-(4.3) thus become obvious, so that since (4.6), (4.9), (4.10) and (2.19) in conjunction with (2.7) imply that $u(\cdot, 0) = u_0$, $u_t(\cdot, 0) = u_{0t}$, $u_{tt}(\cdot, 0) = u_{0tt}$ and $\Theta(\cdot, 0) = \Theta_0$ in Ω , it remains to observe that by (1.5), the identity

$$u_{ttt} = -\alpha u_{tt} + \gamma(\Theta)u_{xxt} + \gamma'(\Theta)\Theta_x u_{xt} + \widehat{\gamma}(\Theta)u_{xx} + \widehat{\gamma}'(\Theta)\Theta_x u_x \quad (4.11)$$

holds in the sense of distributions on $\Omega \times (0, T_*)$. Therefore, namely, (4.1)-(4.3) ensure that u_{ttt} enjoys the regularity property in (4.4), and that (4.11) is actually satisfied at each point in $\Omega \times (0, T_*)$. Since Proposition 1.1 has already asserted that the second sub-problem of (1.1) is satisfied in the classical sense, the proof is complete. \square

The essence of our main result has thus actually been achieved already.

Proof of Theorem 1.1. For initial data satisfying (1.8) and (2.1), we only need to combine the outcome of Lemma 4.1 with a standard extension argument; details for a corresponding construction addressing the closely related scenario of Proposition 1.1 can be found in [17, Proof of Theorem 1.1]. General initial data merely satisfying (1.8) can thereupon readily be dealt with by applying the above to $\tilde{u}_0 := u_0 - y_0$, $\tilde{u}_{0t} := u_{0t} - y_1$ and $\tilde{u}_{0tt} := u_{0tt} - y_2$, with

$$y_0 := \frac{1}{|\Omega|} \int_{\Omega} u_0, \quad y_1 := \frac{1}{|\Omega|} \int_{\Omega} u_{0t}, \quad y_2 := \frac{1}{|\Omega|} \int_{\Omega} u_{0tt},$$

by thereupon letting $(\tilde{u}, \tilde{\Theta})$ denote the corresponding solution of

$$\begin{cases} \tilde{u}_{ttt} + \alpha \tilde{u}_{tt} = (\gamma(\tilde{\Theta})\tilde{u}_{xt})_x + (\widehat{\gamma}(\tilde{\Theta})\tilde{u}_x)_{x'}, & x \in \Omega, \quad t \in (0, T_{\max}), \\ \tilde{\Theta}_t = D\tilde{\Theta}_{xx} + \Gamma(\tilde{\Theta})\tilde{u}_{xt}^2, & x \in \Omega, \quad t \in (0, T_{\max}), \\ \tilde{u}_x = 0, \quad \tilde{\Theta}_x = 0, & x \in \partial\Omega, \quad t \in (0, T_{\max}), \\ (\tilde{u}, \tilde{u}_t, \tilde{u}_{tt}, \tilde{\Theta})(x, 0) = (\tilde{u}_0, \tilde{u}_{0t}, \tilde{u}_{0tt}, \tilde{\Theta}_0)(x), & x \in \Omega, \end{cases}$$

maximally extended up to some $T_{\max} \in (0, \infty]$, and by defining $u(x, t) := \tilde{u}(x, t) + y(t)$ as well as $\Theta(x, t) := \tilde{\Theta}(x, t)$ for $(x, t) \in \bar{\Omega} \times [0, T_{\max})$, where $y \in C^\infty([0, \infty))$ denotes the solution of $y'''(t) + \alpha y''(t) = 0$ in $(0, \infty)$ satisfying $y(0) = y_0$, $y'(0) = y_1$ and $y''(0) = y_2$. \square

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