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Reflected BSDEs Driven by RCLL Martingales with Stochastic Lipschitz Coefficient in a General Filtration: Analysis and Applications

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Abstract. In this paper, we investigate reflected backward stochastic differential equations with a single, discontinuous barrier, driven by a right-continuous, left-limited martingale within a general filtration. We establish the existence and uniqueness of solutions under a stochastic Lipschitz condition on the generator and a reflection process that is right-continuous with left limits. As an application, we use these results to determine fair pricing for American contingent claim options in a financial market driven by Azéma's martingale, incorporating elements of asymmetric information.

AMS subject classifications: 60H05, 60H10, 60H30

Key words: Reflected BSDEs, RCLL martingales, stochastic Lipschitz coefficient, Azéma's martingale market model, American option.

1 Introduction

The notion of one barrier reflected backward stochastic differential equations (RBSDEs) was introduced by El Karoui *et al.* [13] within the framework of a Brow-

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nian filtration associated with a standard Brownian motion $(B_t)_{t\leq T}$. In this framework, a solution to this equation, defined for a horizon time $0<T<\infty$, a coefficient f, a terminal value ξ, and a barrier $L=(L_t)_{t\leq T}$ is represented by a triple of adapted processes $(Y_t,Z_t,K_t)_{t\leq T}$ such that

$$Y_t=\xi+\int_t^T f(s,Y_s,Z_s)ds+(K_T-K_t)-\int_t^T Z_s dB_s, \quad Y_t\geq L_t, \quad \forall t\leq T. \tag{1.1}$$

In other words, the solution to the equation satisfies a certain differential equation and a reflecting boundary condition. In the Eq. (1.1), the non-decreasing continuous process K is introduced to keep the state process Y above the barrier L in a minimal energy. The process K only comes into play when Y reaches the obstacle L, as indicated by the condition $\int_0^T (Y_s-L_s)dK_s=0$. El Karoui *et al.* [13] have shown that when the terminal value ξ is square integrable, the coefficient f is uniformly Lipschitz with respect to (y,z), and the barrier L is continuous, the equation has a unique solution.

The literature on RBSDEs has seen numerous significant contributions. For instance, Hamadène and Lepeltier [26] study the case where the barrier L is right-continuous and left-upper semi-continuous, which implies positive jumps of L, and also explore the applications of RBSDEs in mixed control or zero-sum games. Meanwhile, Matoussi [36] establishes the existence of a solution to Eq. (1.1) when the driver f has linear growth and is only continuous with respect to (y,z). Building on these works, Hamadène [25] investigates discontinuous RBSDEs with a right-continuous and left-limited (RCLL) barrier under the same consideration made in [13], linking the findings with stochastic mixed control problems.

RBSDEs have been studied in various stochastic settings beyond the classical Brownian motion framework. Hamadène and Ouknine [27] considered the case where the filtration is generated by a Brownian motion and an independent Poisson point process. They allowed the obstacle to have only inaccessible jumps and established the existence and uniqueness of the reflected solution under the Lipschitz assumption on the coefficient. This work was later extended in [22,28] to include barriers with general jumps that can be inaccessible or predictable. Other interesting results on RBSDEs can be found in [15,34].

The objective of this paper is to explore the problem of existence and uniqueness for RBSDEs driven by a fairly RCLL general martingale with one reflected RCLL obstacle in an arbitrary filtered probability space. Our work builds upon and expands upon the results presented in previous studies such as [7,12,13,15,22,25–28] (see also [19,32,42] for other related works). However, our approach is more comprehensive and we encounter several challenges in our problem, including:

- Firstly, our filtration is general and must only satisfy the usual assumptions and a quasi-left continuity condition. This is in contrast to previous studies, which require the filtration to be generated by or support a continuous martingale and a random jump measure.
- Secondly, our generator satisfies a stochastic Lipschitz condition, which allows us to handle more flexible financial market models with jumps.
- Finally, the jumps of the obstacles are general and can occur at either predictable or inaccessible stopping times, resulting in both types of jumps in the state process of the solution.

This paper provides a comprehensive overview of the use of RBSDEs in general filtration to value American options between a company (or investors) and an insider on the stock price of a given firm. Such problems arise frequently in real-world economics, and a notable example is Albert Wiggin, who sold his stocks (which can be considered as an American put option) to other investors on the open market, using insider information about his bank, Chase National Bank, to make bets on the stock market. When the market crashed in 1929, Wiggin's bets paid off, and he made a substantial profit of $4 million by selling his stocks at a higher price than he had purchased them for. This situation can be mathematically modeled by considering a market model where the dynamic of the company's stock price is driven by Azéma's martingale, and the filtration is the one that contains the natural flow of information of the public market, i.e. the filtration generated by the Azéma's martingale, as well as the additional information carried by the insider. Emery introduced the solution to the so-called structure equation in [21] and proved that it has the chaotic (particularly the predictable) representation property. Therefore, it is reasonable to study RBSDEs driven by it and to address the problem of the existence of a functional value for American contingent claims in a general financial market driven by this martingale.

The paper is structured as follows. Section 2 provides relevant definitions, notations, assumptions, and properties for our problem. In Section 3, we present our main result on the existence and uniqueness of the solution to reflected BSDEs with one lower reflecting RCLL barrier. To achieve this, we combine optimal stopping theory with a convergence theorem for a monotonic sequence of processes. Finally, in Section 4, we explore the pricing of American options in an insider trading setting within a financial market model driven by Azéma's martingale. We introduce fundamental concepts, definitions, and mathematical modeling of our problem and discuss the existence of a fair value function for these contracts. This function is characterized as the solution to the RBSDE analyzed in Section 3. Finally, we demonstrate the existence of the so called saddle point, provided that the obstacle satisfies additional regularity trajectory assumptions.

2 Preliminaries

Let $(\Omega,\mathscr{F},\mathbb{F},\mathbb{P})$ be a complete filtered probability space where the filtration $\mathbb{F}:=(\mathscr{F}_t)_t$ is quasi-left continuous and satisfies the usual conditions of right-continuity and completeness with $\mathscr{F}_T=\mathscr{F}$. The initial σ-field $\mathscr{F}_0$ is assumed to be trivial. The equality $X=Y$ between any two processes $(X_t)_t$ and $(Y_t)_t$ must be understood in the indistinguishable sense i.e. $\mathbb{P}(\omega:X_t(\omega)=Y_t(\omega),\forall t)=1$. The same signification holds for $X\leq Y$. For a given RCLL process $(Y_t)_t$, $Y_{t-}=\lim_{s\nearrow t}Y_s$ is the left limits of Y at t (by convention, we set $Y_{0-}=Y_0$ and $\mathscr{F}_{0-}=\mathscr{F}_0$). $(Y_-)=(Y_{t-})_t$ is the left limited process and $\Delta Y_t=Y_t-Y_{t-}$ the jump of Y at time t.

Next, for given two locally square integrable $\mathbb{F}$-martingales M and N, we denote by $\langle M,N\rangle$ the predictable $\mathbb{F}$-dual projection of the quadratic co-variation process $[M,N]$, by M^c the continuous part of M. Finally, for a finite variation process $(A_t)_t$, its total variation process would be denoted by $(|A|_t)_t$ and for $x\in\mathbb{R}$, we remember that $x^+=\max(x,0),x^-=(-x)^+=-\min(x,0)$.

Let $T>0$ be a fixed time horizon, we assume given an $\mathbb{R}$-valued, square-integrable $\mathbb{F}$-martingale $M:=(M)_{t\leq T}$ on $(\Omega,\mathscr{F},\mathbb{F}=(\mathscr{F}_t)_{t\leq T},\mathbb{P})$. Since the filtration $\mathbb{F}$ is right-continuous, there exists a modification of M with RCLL paths, hence, we may assume throughout this paper that M is an RCLL process.

We denote by $\mathscr{T}_{\gamma_1}^{\gamma_2}$ the set of $[0,T]$-valued $\mathbb{F}$-stopping times γ such that $\gamma_1\leq\gamma\leq\gamma_2$ a.s. for two $[0,T]$-valued $\mathbb{F}$-stopping times γ_1 and γ_2 such that $\gamma_1\leq\gamma_2$ a.s. and for $\mathscr{F}_t$-progressively measurable RCLL processes $(Y^n)_{n\in\mathbb{N}}$ and Y, we say that $Y^n\to Y$ in UCP (uniformly on compacts in probability) if $\sup_{0\leq s\leq t}|Y^n_s-Y_s|^2\to 0$ in probability $\mathbb{P}$ for every $t>0$.

Remark 2.1. Note that, in one hand, $[M]-\langle M\rangle$ is a uniformly-integrable $\mathbb{F}$-martingale, on the other hand, due to the quasi-left continuity of the filtration $\mathbb{F}$, that is, $\mathscr{F}_\sigma=\mathscr{F}_{\sigma-}$ for every $\mathbb{F}$-predictable stopping time σ, then, the square-integrable $\mathbb{F}$-martingales jump only at $\mathbb{F}$-totally inaccessible stopping times and the $\mathscr{F}_t$-predictable, increasing process $\langle M\rangle$ has continuous paths (see [44, Chapter IV] for more details).

In the current paper, we aim to explore on $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\leq T},\mathbb{P})$ a reflected backward stochastic differential equation driven by the square integrable RCLL martingale of the following form:

$$\begin{aligned} Y_t=\xi+&\int_t^T f(s,Y_s,Z_s)d\langle M\rangle_s+(K_T-K_t)\\ &-\int_t^T Z_s dM_s-\int_t^T dN_s,\quad 0\leq t\leq T. \end{aligned} \tag{2.1a}$$

$$Y_t \geq L_t, \quad 0 \leq t \leq T. \tag{2.1b}$$

If K^c is the continuous part of K, then $\int_0^T (Y_t - L_t) dK_t^c = 0$. (2.1c)

If K^d is the purely discontinuous part of K, then K^d is predictable and $K_t^d = \sum_{0<s\leq t} (Y_s - L_{s-})^-$. (2.1d)

Next, we introduce the following processes and spaces to describe the parameters and the solution of the Eq. (2.1).

Let $\beta > 0, (\alpha_t)_{t\leq T}$ a non-negative $\mathscr{F}_t$-adapted process and $(A_t)_{t\leq T}$ be the increasing continuous process defined as $A_t := \int_0^t \alpha_s^2 d\langle M\rangle_s$ for any $t \leq T$ and we introduce the following spaces:

- $\mathscr{S}^2$: the space of one-dimensional $\mathscr{F}_t$-predictable RCLL increasing processes $(K_t)_{t\leq T}$ such that

$$\| K \|_{\mathscr{S}^2}^2 = \mathbb{E}\left[\sup_{0\leq t\leq T} |K_t|^2\right] < \infty.$$

- $\mathscr{C}_\beta^2$: the space of $\mathbb{R}$-valued $\mathscr{F}_t$-progressively measurable processes $(F_t)_{t\leq T}$ such that

$$\| F \|_{\mathscr{C}_\beta^2}^2 = \mathbb{E}\left[\int_0^T e^{\beta A_s} |F_s|^2 ds\right] < \infty,$$

  by convention we set $\mathscr{C}^2 := \mathscr{C}_0^2$.
- $\mathbb{L}^2(\mathscr{F}_t)$: the set of one-dimensional $\mathscr{F}_t$-measurable square-integrable random variables ξ.
- $\mathbb{L}_\beta^2$: the set of one-dimensional $\mathscr{F}_T$-measurable random variables ξ such that

$$\|\xi\|_\beta^2 := \mathbb{E}\left[e^{\beta A_T} |\xi|^2\right] < \infty.$$

- $\mathscr{S}_\beta^2$: the space of one-dimensional $\mathscr{F}_t$-adapted RCLL processes $(Y_t)_{t\leq T}$ such that

$$\| Y \|_{\mathscr{S}_\beta^2}^2 = \mathbb{E}\left[\sup_{0\leq t\leq T} e^{\beta A_t} |Y_t|^2\right] < \infty.$$

- $\mathscr{S}_\beta^{2,\alpha}$: the space of one-dimensional $\mathscr{F}_t$-adapted RCLL processes $(Y_t)_{t\leq T}$ such that

$$\| Y \|_{\mathscr{S}_\beta^2}^2 = \mathbb{E}\left[\int_0^T e^{\beta A_s} |\alpha_s Y_s|^2 d\langle M\rangle_s\right] < \infty.$$

- $\mathscr{H}_\beta^2$: the space of $\mathbb{R}$-valued $\mathscr{F}_t$-predictable processes $(Z_t)_{t\leq T}$ such that

$$\| Z \|^2_{\mathscr{H}_\beta^2} = \mathbb{E}\left[\int_0^T e^{\beta A_s}|Z_s|^2 d\langle M\rangle_s\right] < \infty,$$

  by convention we set $\mathscr{H}^2 := \mathscr{H}_0^2$.
- $\mathscr{M}_\beta^2$: the space of one dimensional square-integrable $\mathbb{F}$-martingale $(N_t)_{t\leq T}$ orthogonal to M such that

$$\| N \|^2_{\mathscr{M}_\beta^2} = \mathbb{E}\left[\int_0^T e^{\beta A_s} d[N]_s\right] < \infty,$$

  by convention we set $\mathscr{M}^2 := \mathscr{M}_0^2$.
- $\mathfrak{D}_\beta^2 = \left(\mathscr{S}_\beta^2 \cap \mathscr{S}_\beta^{2,\alpha}\right) \times \mathscr{H}_\beta^2 \times \mathscr{S}^2 \times \mathscr{M}_\beta^2$.

Remark 2.2. If the underlying filtration $\mathbb{F}$ is not quasi-left continuous, the sharp bracket $\langle M\rangle$, may not be continuous in general. Instead, we may assume as in [39, Assumption 2.1 and Remark 2.1, p. 3], in the case of an n-dimensional, square-integrable martingale $M_t = \{(M_t^1, M_t^2, \ldots, M_t^n)^*, 0 \leq t \leq T\}$, that $d\langle M\rangle_t = m_t m_t^\star dQ_t$, where Q is a bounded, $\mathscr{F}_t$-adapted, continuous, increasing processes with $Q_0 = 0$, and $(m_t)_{t\leq T}$ an $\mathbb{R}^{n\times n}$-dimensional, $\mathscr{F}_t$-predictable process. Moreover, we may assume that m is a symmetric $\mathbb{R}^{n\times n}$-matrix.

Hypothesis on the data of the RBSDE (2.1)**.** The triplet (ξ, f, L) is such that:

(H1) The terminal variable $\xi \in \mathbb{L}_\beta^2$.

(H2) The driver $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is such that:

(i) For all (y,z) the stochastic process $f(\cdot,y,z)$ is $\mathscr{F}_t$-progressively measurable.

(ii) There exists two non-negative $\mathscr{F}_t$-adapted processes $(\kappa_t)_{t\leq T}$ and $(\gamma_t)_{t\leq T}$ such that

- $*$ for all $t \in [0,T]$, y, $y' \in \mathbb{R}$ and $z, z' \in \mathbb{R}$,

$$|f(t,y,z) - f(t,y',z')| \leq \kappa_t |y - y'| + \gamma_t |z - z'|,$$

- $*$ there exists $\epsilon > 0$ such that $\alpha_s^2 := \kappa_s + \gamma_s^2 \geq \epsilon$ and $f(\cdot,0,0)/(\alpha.) \in \mathscr{H}_\beta^2$.

(H3) The barrier $(L_t)_{t\leq T}$ is an real-valued $\mathscr{F}_t$-progressively measurable RCLL processes satisfying

(i) $\xi \geq L_T$ $\mathbb{P}$-a.s.,

(ii) $\mathbb{E}[\sup_{0\leq t\leq T} e^{2\beta A_t}|L_t^+|^2]<\infty$.

Definition 2.1. *Let $\beta>0$ and $(\alpha_t)_{t\leq T}$ a non negative $\mathscr{F}_t$-adapted process. A solution to the RBSDE* (2.1) *with jumps associated with parameters (ξ,f,L) is a quintuplet of processes (Y,Z,K,N) satisfying* (2.1) *and belongs to $\mathfrak{D}^2_\beta$.*

According to our definition, the process Y's jumping times are derived from both those of the barrier L (predictable jumps) and those of its martingale part process (inaccessible jumps).

Remark 2.3. (i) Recall that for any pair of $\mathbb{F}$-semimartingales S^1 and S^2, the operation $(S^1,S^2)\to[S^1,S^2]$ is bilinear and symmetric. Therefore, we have the following polarization identities:

$$[S^1+S^2]=[S^1]+2[S^1,S^2]+[S^2],$$
$$[S^1-S^2]=[S^1]-2[S^1,S^2]+[S^2].$$

Applying the second identity to the dynamic of the process $(Y_t)_{t\leq T}$ given by (2.1a) with

$$(S^1,S^2)=\left(\int_0^\cdot Z_s dM_s+N,\int_0^\cdot f(s,Y_s,Z_s)d\langle M\rangle_s+K-Y_0\right),$$

the first to S^1 and taking into account the continuity of the process $\langle M\rangle$, the orthogonality between N and M arises from the property of martingale representation, and [30, Proposition 4.50(d), p. 53], yields to

$$[Y]=\sum_{0<s\leq\cdot}\left(\Delta K_s^d\right)^2+\int_0^\cdot |Z_s|^2 d[M]_s+\int_0^\cdot d[N]_s.$$

Thus, the jump part of the process $[Y]$ is described by

$$\sum_{0<s\leq\cdot}(\Delta Y_s)^2=\sum_{0<s\leq\cdot}\left(\Delta K_s^d\right)^2+\sum_{0<s\leq\cdot}|Z_s|^2(\Delta M_s)^2+\sum_{0<s\leq\cdot}(\Delta N_s)^2, \tag{2.2}$$

and the path-by-path continuous part of $t\mapsto[Y]_t$ is given by

$$[Y]^c=\int_0^\cdot |Z_s|^2 d\langle M^c\rangle_s+\int_0^\cdot d\langle N^c\rangle_s.$$

(For such a path-wise decomposition for the quadratic variation, the reader is referred to [44, p. 70]).

(ii) For any process Z that belongs to $\mathscr{H}^2$, the process $(\int_0^{\cdot} Z_s dM_s)^2 - \int_0^{\cdot} |Z_s|^2 d[M]_s$ is an $\mathbb{F}$-martingale [44, Theorem 27, p. 71], and we have

$$\mathbb{E}^{\mathscr{F}_t}\left[\left(\int_t^T Z_s dM_s\right)^2\right] \\ = \mathbb{E}^{\mathscr{F}_t}\left[\int_t^T |Z_s|^2 d[M]_s\right] = \mathbb{E}^{\mathscr{F}_t}\left[\int_t^T |Z_s|^2 d\langle M\rangle_s\right].$$

(iii) Using the sharp bracket version of the Kunita-Watanabe inequality [44, p. 148], for a jointly measurable process A such that $A/\alpha \in \mathscr{H}_b^2$ for some $b>0$, we have

$$\left(\int_{t_1}^{t_2} \chi_s d\langle M\rangle_s\right) \leq \frac{1}{\sqrt{b}}\left(e^{-bA_{t_1}} - e^{-bA_{t_2}}\right)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} e^{bA_s}\left|\frac{\chi_s}{\alpha_s}\right|^2 d\langle M\rangle_s\right)^{\frac{1}{2}}. \tag{2.3}$$

(iv) We point out that, since $([M,N]-\langle M,N\rangle)$ is a martingale (see [30, Proposition 4.50(b), p. 53]). If Z is an element of $\mathscr{H}_\beta^2$, then

$$\mathbb{E}^{\mathscr{F}_t}\left[\int_t^T e^{\beta A_s} Z_s d[M,N]_s\right] = \mathbb{E}\left[\int_t^T e^{\beta A_s} Z_s d\langle M,N\rangle_s\right]. \tag{2.4}$$

It should be noted that the last term is equal to zero when M and N are orthogonal and

$$\mathbb{E}^{\mathscr{F}_t}\left[\int_t^T e^{\beta A_s}|Z_s|^2 d[M]_s\right] = \mathbb{E}^{\mathscr{F}_t}\left[\int_t^T e^{\beta A_s}|Z_s|^2 d\langle M\rangle_s\right]. \tag{2.5}$$

Remark 2.4 (Positioning Relative to Recent Work)**.** To place our results in context, we clarify the positioning of our contribution with respect to the related works [16, 17, 20]. These papers by the present authors are subsequent extensions of the results developed here to the doubly reflected setting (two RCLL barriers), where additional issues arise, such as the interaction between obstacles and a Dynkin-game interpretation. By contrast, the present manuscript develops the foundational framework for the singly reflected case (one lower RCLL barrier) under general assumptions, establishing existence, uniqueness, and the key a priori estimates on which [16,17,20] build. In addition, the paper develops a coherent insider-type financial problem that motivates the analysis from the formulation of the RBSDE and continues through the application in Section 4, thereby showing how the abstract results of the next Section 3 can be implemented in a concrete setting.

3 BSDEs with one lower reflecting RCLL barrier

3.1 Uniqueness result

Let now (ξ,f,L) and (ξ',f',L') be two sets of data, each satisfying the above assumptions **(H1)-(H3)**. Let $(Y_t,Z_t,K_t,N_t)_{t\leq T}$ (respectively $(Y'_t,Z'_t,K'_t,N'_t)_{t\leq T}$) denote a solution of the RBSDE (2.1) with data (ξ,f,L) (respectively (ξ',f',L')) in the sense of Definition 2.1. The following proposition will be useful for further applications.

Proposition 3.1. *Assume given two $\mathbb{F}$-stopping times τ,σ such that $\tau\in\mathscr{T}_0^T$ and $\sigma\in\mathscr{T}_\tau^T$. Then, for any $\beta>2$, there exists a constant $C_\beta>0$ such that*

$$
\begin{aligned}
&\mathbb{E}^{\mathscr{F}_\tau}\left[\sup_{\tau\leq t\leq\sigma} e^{\beta A_t}|Y_t-Y'_t|^2\right]+\mathbb{E}^{\mathscr{F}_\tau}\left[\int_\tau^\sigma e^{\beta A_s}\alpha_s^2|Y_s-Y'_s|^2 d\langle M\rangle_s\right]\\
&\quad+\mathbb{E}^{\mathscr{F}_\tau}\left[\int_\tau^\sigma e^{\beta A_s}|Z_s-Z'_s|^2 d\langle M\rangle_s\right]+\mathbb{E}^{\mathscr{F}_\tau}\left[\int_\tau^\sigma e^{\beta A_s} d[N-N']_s\right]\\
&\leq C_\beta\Bigg\{\mathbb{E}^{\mathscr{F}_\tau}\left[e^{\beta A_\sigma}|Y_\sigma-Y'_\sigma|^2\right]+\mathbb{E}^{\mathscr{F}_\tau}\left[\int_\tau^\sigma e^{\beta A_s}\left|\frac{f(s,Y'_s,Z'_s)-f'(s,Y'_s,Z'_s)}{\alpha_s}\right|^2 d\langle M\rangle_s\right]\\
&\quad+\mathbb{E}^{\mathscr{F}_\tau}\left[\int_\tau^\sigma e^{\beta A_s}\left((L_{s-}-L'_{s-})^+dK_s+(L'_{s-}-L_{s-})^+dK'_s\right)\right]\Bigg\}.
\end{aligned}
$$

Proof. Using Itô's formula [44, Theorem 33, p. 81], we can write for $\beta>0$ and for any $\tau\leq t\leq\sigma$,

$$
\begin{aligned}
&e^{\beta A_t}|Y_t-Y'_t|^2+\beta\int_t^\sigma e^{\beta A_s}|Y_s-Y'_s|^2 d\langle M\rangle_s+\int_t^\sigma e^{\beta A_s}|Z_s-Z'_s|^2 d\langle M\rangle_s\\
&\quad+2\int_t^\sigma e^{\beta A_s}|Z_s-Z'_s|^2 d\langle M,N\rangle_s^c+\int_t^\sigma e^{2A_s} d\langle N\rangle_s^c+\sum_{t<s\leq\sigma} e^{\beta A_s}(\Delta Y_s)^2\\
&=e^{\beta A_\sigma}|Y_\sigma-Y'_\sigma|^2+2\int_t^\sigma e^{\beta A_s}(Y_s-Y'_s)\left(f(s,Y_s,Z_s)-f'(s,Y'_s,Z'_s)\right)d\langle M\rangle_s\\
&\quad+2\int_t^\sigma e^{\beta A_s}(Y_{s-}-Y'_{s-})(dK_s-dK'_s)\\
&\quad-2\int_t^\sigma e^{\beta A_s}(Y_{s-}-Y'_{s-})(Z_s-Z'_s)dM_s-2\int_t^\sigma e^{\beta A_s}(Y_{s-}-Y'_{s-})(dN_s-dN'_s). \qquad (3.1)
\end{aligned}
$$

By observing the integrability condition that the solution of BSDE (2.1) satisfies, we can infer that the two terms located in the final line of (3.1) are uniformly integrable $\mathbb{F}$-martingales. Moreover, assumption **(H2)**(ii) and inequality $2ab \leq \theta a^2 + b^2/\theta$, yields for any $\theta > 0$,

$$\begin{aligned}&2(Y_s - Y'_s)\big(f(s,Y_s,Z_s) - f(s,Y'_s,Z'_s)\big)\\ \leq& 2\kappa_s|Y_s - Y'_s|^2 + 2\gamma_s|Y_s - Y'_s||Z_s - Z'_s|\\ \leq& 2\alpha_s^2|Y_s - Y'_s|^2 + \frac{1}{2}|Z_s - Z'_s|^2. \end{aligned} \tag{3.2}$$

On the other hand,

$$\begin{aligned}&2(Y_s - Y'_s)\big(f(s,Y'_s,Z'_s) - f'(s,Y'_s,Z'_s)\big)\\ \leq& \theta\alpha_s^2|Y_s - Y'_s|^2 + \frac{1}{\theta}\left|\frac{f(s,Y'_s,Z'_s) - f'(s,Y'_s,Z'_s)}{\alpha_s}\right|^2. \end{aligned} \tag{3.3}$$

Note that the process $(|(f(s,Y'_s,Z'_s) - f'(s,Y'_s,Z'_s))/\alpha_s|^2)_{s\leq T}$ satisfies the integrability property of processes belonging to the space $\mathscr{H}_\beta^2$. Indeed, a simple calculation yields to

$$\begin{aligned}&\left|\frac{f(s,Y'_s,Z'_s) - f'(s,Y'_s,Z'_s)}{\alpha_s}\right|^2\\ \leq& 6\big(\alpha_s^2|Y_s|^2 + \|Z_s\|^2 + \alpha_s^2|Y'_s|^2 + \|Z'_s\|^2\big) + 6\left(\left|\frac{f(s,0,0)}{\alpha_s}\right|^2 + \left|\frac{f'(s,0,0)}{\alpha_s}\right|^2\right).\end{aligned}$$

Since $(Y,Y',Z,Z') \in (\mathscr{S}_\beta^{2,\alpha})^2 \times (\mathscr{H}_\beta^2)^2$ and $f(\cdot,0,0), f'(\cdot,0,0) \in \mathscr{H}_\beta^2$, we deduce that

$$\mathbb{E}\left[\int_0^T e^{\beta A_s}\left|\frac{f(s,Y'_s,Z'_s) - f'(s,Y'_s,Z'_s)}{\alpha_s}\right|^2 d\langle M\rangle_s\right] < \infty.$$

By contrast, we can derive, through the utilization of the Skorohod condition (2.1c), the following:

$$\begin{aligned}&\int_t^\sigma e^{\beta A_s}\big(Y_{s-} - Y'_{s-}\big)\big(dK_s - dK'_s\big)\\ =& \int_t^\sigma e^{\beta A_s}(Y_{s-} - L_{s-})dK_s^+ + \int_t^\sigma e^{\beta A_s}(L_{s-} - Y_{s-})dK'_s\\ &+ \int_t^\sigma e^{\beta A_s}(L'_{s-} - Y'_{s-})dK_s + \int_t^\sigma e^{\beta A_s}(Y'_{s-} - L'_{s-})dK'_s + \int_t^\sigma (L_{s-} - L'_{s-})(dK_s - dK'_s)\\ \leq& \int_t^\sigma \big(L_{s-} - L'_{s-}\big)\big(dK_s - dK'_s\big).\end{aligned} \tag{3.4}$$

After incorporating relations (2.2), (2.4), (2.5), (3.2)-(3.4) into Eq. (3.1) after computing the conditional expectation on both sides, we can derive

$$
\begin{aligned}
&e^{\beta A_t}|Y_t-Y'_t|^2+\big(\beta-(\sigma+2)\big)\mathbb{E}^{\mathscr{F}_t}\left[\int_t^\sigma e^{\beta A_s}\alpha_s^2|Y_s-Y'_s|^2 d\langle M\rangle_s\right]\\
&\quad+\frac{1}{2}\mathbb{E}^{\mathscr{F}_t}\left[\int_t^\sigma e^{\beta A_s}|Z_s-Z'_s|^2 d\langle M\rangle_s\right]+\mathbb{E}^{\mathscr{F}_t}\left[\int_t^\sigma e^{\beta A_s}d[N-N']_s\right]\\
&\leq \mathbb{E}^{\mathscr{F}_t}\left[e^{\beta A_\sigma}|Y_\sigma-Y'_\sigma|^2\right]+\frac{1}{\theta}\mathbb{E}^{\mathscr{F}_t}\left[\int_t^\sigma e^{\beta A_s}\left|\frac{f(s,Y'_s,Z'_s)-f'(s,Y'_s,Z'_s)}{\alpha_s}\right|^2 d\langle M\rangle_s\right]\\
&\quad+2\mathbb{E}^{\mathscr{F}_t}\left[\int_t^\sigma e^{\beta A_s}\big((L_{s-}-L'_{s-})^+dK_s+(L'_{s-}-L_{s-})^+dK'_s\big)\right].
\end{aligned}
$$

Next, we choose $\sigma>0$ such that $\beta>\theta+2$, and then take the conditional expectation with respect to $\mathscr{F}_\tau$ to obtain

$$
\begin{aligned}
&\sup_{\tau\leq t\leq\sigma}\mathbb{E}^{\mathscr{F}_\tau}\left[e^{\beta A_t}|Y_t-Y'_t|^2\right]+\mathbb{E}^{\mathscr{F}_\tau}\left[\int_\tau^\sigma e^{\beta A_s}\alpha_s^2|Y_s-Y'_s|^2 d\langle M\rangle_s\right]\\
&\quad+\mathbb{E}^{\mathscr{F}_\tau}\left[\int_\tau^\sigma e^{\beta A_s}|Z_s-Z'_s|^2 d\langle M\rangle_s\right]+\mathbb{E}^{\mathscr{F}_\tau}\left[\int_\tau^\sigma e^{\beta A_s}d[N-N']_s\right]\\
&\leq C_\beta\Bigg\{\mathbb{E}^{\mathscr{F}_\tau}\left[e^{\beta A_\sigma}|Y_\sigma-Y'_\sigma|^2\right]+\mathbb{E}^{\mathscr{F}_\tau}\left[\int_\tau^\sigma e^{\beta A_s}\left|\frac{f(s,Y'_s,Z'_s)-f'(s,Y'_s,Z'_s)}{\alpha_s}\right|^2 d\langle M\rangle_s\right]\\
&\quad+\mathbb{E}^{\mathscr{F}_\tau}\left[\int_\tau^\sigma e^{\beta A_s}\big((L_{s-}-L'_{s-})^+dK_s+(L'_{s-}-L_{s-})^+dK'_s\big)\right]\Bigg\}.
\end{aligned}
\tag{3.5}
$$

By contrast, by combining Eq. (3.1) with the conditional expectation taken with respect to $\mathscr{F}_t$, along with (2.5) and (2.4), for all t in the interval $[\tau,\sigma]$, $\mathbb{P}$-a.s., we obtain

$$
\begin{aligned}
&e^{\beta A_t}|Y_t-Y'_t|^2\\
&\leq \mathbb{E}^{\mathscr{F}_t}\left[e^{\beta A_T}|\xi-\xi'|^2\right]+\frac{1}{\sigma}\mathbb{E}^{\mathscr{F}_t}\left[\int_\tau^\sigma e^{\beta A_s}\left|\frac{f(s,Y'_s,Z'_s)-f'(s,Y'_s,Z'_s)}{\alpha_s}\right|^2 d\langle M\rangle_s\right]\\
&\quad+\mathbb{E}^{\mathscr{F}_t}\left[\int_\tau^\sigma e^{\beta A_s}\big((L_{s-}-L'_{s-})^+dK_s^++(L'_{s-}-L_{s-})^+dK'^{,+}_s\big)\right]\\
&\quad+2\mathbb{E}^{\mathscr{F}_t}\left[\sup_{\tau\leq t\leq\sigma}\left|\int_t^\sigma e^{\beta A_s}(Y_{s-}-Y'_{s-})(Z_s-Z'_s)dM_s\right|\right]\\
&\quad+2\mathbb{E}^{\mathscr{F}_t}\left[\sup_{\tau\leq t\leq\sigma}\left|\int_t^\sigma e^{\beta A_s}(Y_{s-}-Y'_{s-})(dN_s-dN'_s)\right|\right].
\end{aligned}
\tag{3.6}
$$

Using the Burkholder-Davis-Gundy's (BDG) inequality, we have

$$
\begin{aligned}
&2\mathbb{E}^{\mathscr{F}_\tau}\left[\sup_{\tau\le t\le T}\left|\int_t^\sigma e^{\beta A_s}(Y_{s-}-Y'_{s-})(Z_s-Z'_s)dM_s\right|\right]\\
\le&2c\mathbb{E}^{\mathscr{F}_\tau}\left[\left(\int_\tau^\sigma e^{2\beta A_s}|Y_{s-}-Y'_{s-}|^2|Z_s-Z'_s|^2d[M]_s\right)^{\frac{1}{2}}\right]\\
\le&\frac{1}{4}\mathbb{E}^{\mathscr{F}_\tau}\left[\sup_{\tau\le t\le\sigma}e^{\beta A_t}|Y_t-Y'_t|^2\right]+4c^2\mathbb{E}^{\mathscr{F}_\tau}\left[\int_\tau^\sigma e^{\beta A_s}|Z_s-Z'_s|^2d\langle M\rangle_s\right].
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
&2\mathbb{E}^{\mathscr{F}_\tau}\left[\sup_{\tau\le t\le\sigma}\left|\int_t^T e^{\beta A_s}(Y_{s-}-Y'_{s-})d(N_s-N'_s)\right|\right]\\
\le&2c\mathbb{E}^{\mathscr{F}_\tau}\left[\left(\int_\tau^\sigma e^{2\beta A_s}|Y_{s-}-Y'_{s-}|^2d[N-N']_s\right)^{\frac{1}{2}}\right]\\
\le&\frac{1}{4}\mathbb{E}^{\mathscr{F}_\tau}\left[\sup_{\tau\le t\le\sigma}e^{\beta A_t}|Y_t-Y'_t|^2\right]+4c^2\mathbb{E}^{\mathscr{F}_\tau}\left[\int_\tau^\sigma e^{\beta A_s}d[N-N']_s\right].
\end{aligned}
$$

After once more computing the conditional expectation, this time with respect to $\mathscr{F}_\tau$, for both sides of Eq. (3.6) and subsequently selecting the same σ as previously determined by (3.5), we get

$$
\begin{aligned}
&\mathbb{E}^{\mathscr{F}_\tau}\left[\sup_{\tau\le t\le\sigma}e^{\beta A_t}|Y_t-Y'_t|^2\right]\\
\le&C_\beta\bigg\{\mathbb{E}^{\mathscr{F}_\tau}\left[e^{\beta A_\sigma}|Y_\sigma-Y'_\sigma|^2\right]+\mathbb{E}^{\mathscr{F}_\tau}\left[\int_\tau^\sigma e^{\beta A_s}\frac{|f(s,Y'_s,Z'_s)-f'(s,Y'_s,Z'_s)|^2}{\alpha_s^2}d\langle M\rangle_s\right]\\
&\quad+\mathbb{E}^{\mathscr{F}_\tau}\left[\int_\tau^\sigma e^{\beta A_s}\big((L_{s-}-L'_{s-})^+dK_s+(L'_{s-}-L_{s-})^+dK'_s\big)\right]\bigg\}.
\end{aligned}
$$

This completes the proof of Proposition 3.1. □

Corollary 3.1. *Assume **(H1)-(H3)**. Then there exists at most one solution* (Y,Z,K,N) *of the RBSDE* (2.1) *associated with* (ξ,f,L)*.*

3.2 Existence result

The main result of the current section is given as follows.

Theorem 3.1. *Under assumptions **(H1)-(H3)**, the RBSDE associated with (ξ,f,L) admit a unique solution $(Y,Z,K,N)\in\mathfrak{D}^2_\beta$ such that*

$$\begin{aligned}&\mathbb{E}\left[\sup_{0\le t\le T} e^{\beta A_t}|Y_t|^2\right]+\mathbb{E}\left[\int_0^T e^{\beta A_s}\alpha_s^2|Y_s|^2 d\langle M\rangle_s\right]+\mathbb{E}\left[|K_T|^2\right]\\&\quad+\mathbb{E}\left[\int_0^T e^{\beta A_s}|Z_s|^2 d\langle M\rangle_s\right]+\mathbb{E}\left[\int_0^T e^{\beta A_s} d[N]_s\right]\\&\le C_\beta\left\{\mathbb{E}\left[e^{\beta A_T}|\xi|^2\right]+\mathbb{E}\left[\int_0^T e^{\beta A_s}\left|\frac{f(s,0,0)}{\alpha_s}\right|^2 d\langle M\rangle_s\right]+\mathbb{E}\left[\sup_{0\le t\le T} e^{2\beta A_t}|L_t^+|^2\right]\right\},\end{aligned}$$

and the state process $(Y_t)_{t\le T}$ can be described as follows:

$$Y_\sigma=\operatorname*{ess\,sup}_{\tau\in\mathscr{T}_\sigma^T}\mathbb{E}^{\mathscr{F}_\sigma}\left[\int_\sigma^\tau g(s,Y_s,Z_s)d\langle M\rangle_s+L_\tau\mathbb{1}_{\{\tau<T\}}+\xi\mathbb{1}_{\{\tau=T\}}\right].$$

Part 1: The coefficient f does not depend on (y,z).

Let us begin by assuming that the function f is not a function of (y,z), that is, for all t, y and z $\mathbb{P}$-a.s., the equality $f(t,y,z)=g(t)$ holds true. In the forthcoming outcome, we will demonstrate the existence of a solution to the RDBSDE that corresponds to the set (ξ,g,L). Namely, the first main result of the current proof is the following.

Theorem 3.2. *The RBSDE associated with (ξ,g,L) admit a unique solution $(Y,Z,K,N)\in\mathfrak{D}^2_\beta$ such that*

$$\begin{aligned}&\mathbb{E}\left[\sup_{0\le t\le T} e^{\beta A_t}|Y_t|^2\right]+\mathbb{E}\left[\int_0^T e^{\beta A_s}\alpha_s^2|Y_s|^2 d\langle M\rangle_s\right]+\mathbb{E}\left[|K_T|^2\right]\\&\quad+\mathbb{E}\left[\int_0^T e^{\beta A_s}|Z_s|^2 d\langle M\rangle_s\right]+\mathbb{E}\left[\int_0^T e^{\beta A_s} d[N]_s\right]\\&\le C_\beta\left\{\mathbb{E}\left[e^{\beta A_T}|\xi|^2\right]+\mathbb{E}\left[\int_0^T e^{\beta A_s}\left|\frac{g(s)}{\alpha_s}\right|^2 d\langle M\rangle_s\right]+\mathbb{E}\left[\sup_{0\le t\le T} e^{2\beta A_t}|L_t^+|^2\right]\right\}.\end{aligned}$$

Furthermore, the state process $(Y_t)_{t\le T}$ can be defined by utilizing the Snell envelope of processes in the following manner:

$$Y_\sigma=\operatorname*{ess\,sup}_{\tau\in\mathscr{T}_\sigma^T}\mathbb{E}\left[\int_\sigma^\tau g(s)d\langle M\rangle_s+S_\tau\mathbb{1}_{\{\tau<T\}}+\xi\mathbb{1}_{\{\tau=T\}}\,|\,\mathscr{F}_\sigma\right]. \tag{3.7}$$

Proof. The proof of Theorem 3.2 will be established in five steps by combining two key components: the theory of optimal stopping and a convergence theorem for a monotonic sequence of processes. These tools enable us to determine whether a solution exists and, if so, to explicitly find it using the formula (3.7).

Step 1: Construction of penalized BSDE and uniform estimation.

Let us begin with the sequence of penalized versions

$$\begin{aligned} Y_t^n = & \xi + \int_t^T g(s) d\langle M\rangle_s + n\int_t^T \left(Y_s^n - L_s\right)^- ds \\ & - \int_t^T Z_s^n dM_s - \int_t^T dN_s^n, \quad 0\le t\le T, \quad n\in\mathbb{N}. \end{aligned} \tag{3.8}$$

We denote $K_t^n := n\int_0^t (Y_s^n - L_s)^- ds$.

Based on the existence and uniqueness results provided in the Appendix 4.2.2, the BSDE (3.8) possesses a unique solution

$$(Y^n, Z^n, N^n) \in \left(\mathscr{S}_\beta^2 \cap \mathscr{S}_\beta^{2,\alpha} \cap \mathscr{C}_\beta^2\right) \times \mathscr{H}_\beta^2 \times \mathscr{M}_\beta^2.$$

We first need some uniform estimation for the sequence of process $\{Y^n, Z^n, K^n, N^n\}_{n\in\mathbb{N}}$.

Lemma 3.1. *For any $\beta > 0$, there exists a constant $C_\beta > 0$ independent of n such that*

$$\begin{aligned} \sup_{n\in\mathbb{N}} \Bigg\{ & \mathbb{E}\left[\sup_{0\le t\le T} e^{\beta A_t}|Y_t^n|^2\right] + \mathbb{E}\left[\int_0^T e^{\beta A_s}\alpha_s^2 |Y_s^n|^2 d\langle M\rangle_s\right] + \mathbb{E}\left[|K_T^n|^2\right] \\ & + \mathbb{E}\left[\int_0^T e^{\beta A_s}|Z_s^n|^2 d\langle M\rangle_s\right] + \mathbb{E}\left[\int_0^T e^{\beta A_s} d[N^n]_s\right]\Bigg\} \\ \le C_\beta \Bigg\{ & \mathbb{E}\left[e^{\beta A_T}|\xi|^2\right] + \mathbb{E}\left[\int_0^T e^{\beta A_s}\left|\frac{g(s)}{\alpha_s}\right|^2 d\langle M\rangle_s\right] + \mathbb{E}\left[\sup_{0\le t\le T} e^{2\beta A_t}|L_t^+|^2\right]\Bigg\} =: \mathfrak{C}_\beta. \end{aligned} \tag{3.9}$$

Proof. For $n\ge 0$, using Itô's formula we have

$$\begin{aligned} & e^{\beta A_t}|Y_t^n|^2 + \beta\int_t^T e^{\beta A_s}|Y_s^n|^2 dA_s + \int_t^T e^{\beta A_s}|Z_s^n|^2 d\langle M\rangle_s \\ & \quad + \int_t^T e^{\beta A_s} d\langle N^n\rangle_s^c + \int_t^T e^{\beta A_s} Z_s^n d\langle M, N^n\rangle_s^c + \sum_{t<s\le T} e^{\beta A_s}\left(\Delta Y_s^n\right)^2 \\ & = e^{\beta A_T}|\xi|^2 + 2\int_t^T e^{\beta A_s} Y_s^n g(s) d\langle M\rangle_s + 2n\int_t^T e^{\beta A_s} Y_s^n \left(Y_s^n - L_s\right)^- ds \end{aligned}$$

$$-2\int_t^T e^{\beta A_s} Y_{s-}^n Z_s^n dM_s - 2\int_t^T e^{\beta A_s} Y_{s-}^n dN_s^n. \tag{3.10}$$

Taking the expectation on both sides, next using equalities (2.2), (2.4), (2.5), inequalities $a(a-b)^- \le b^+(a-b)^-, 2ab \le a^2/\epsilon + \epsilon b^2$ for any $\epsilon > 0$, sharp bracket version of the Kunita-Watanabe inequality, and the fact that stochastic integrals appearing on the right hand of (3.10) are all square-integrable $\mathbb{F}$-martingales, yields

$$\begin{aligned}
&\frac{\beta}{2}\mathbb{E}\left[\int_t^T e^{\beta A_s}|Y_s^n\alpha_s|^2 d\langle M\rangle_s\right] + \mathbb{E}\left[\int_t^T e^{\beta A_s}|Z_s^n|^2 d\langle M\rangle_s\right] + \mathbb{E}\left[\int_t^T e^{\beta A_s} d[N^n]_s\right] \\
&\le \mathbb{E}\left[e^{\beta A_T}|\xi|^2\right] + \frac{2}{\beta}\mathbb{E}\left[\int_0^T e^{\beta A_s}\left|\frac{g(s)}{\alpha_s}\right|^2 d\langle M\rangle_s\right] \\
&\quad + \epsilon\mathbb{E}\left[\sup_{0\le t\le T} e^{2\beta A_t}(L_t^+)^2\right] + \frac{1}{\epsilon}\mathbb{E}\left[|K_T^n - K_t^n|^2\right].
\end{aligned} \tag{3.11}$$

Furthermore, since the process $(\int_0^\cdot Z_s^n dM_s)(\int_0^\cdot dN_s^n) - \int_0^\cdot Z_s^n d[M,N^n]_s$ is a uniformly integrable martingale starting from zero [30, Proposition 4.50(a), p. 53], the isometric formula and the orthogonality property (2.4), implies

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_t^T Z_s^n dM_s\right)\left(\int_t^T dN_s^n\right)\right] \\
&= \mathbb{E}\left[\int_t^T Z_s^n d[M,N^n]_s\right] = \mathbb{E}\left[\int_t^T Z_s^n d\langle M,N^n\rangle_s\right] = 0.
\end{aligned} \tag{3.12}$$

From Eqs. (3.8), by squaring and taking into account (3.12) and inequality (2.3), we conclude that

$$\begin{aligned}
&\mathbb{E}\left[|K_T^n - K_t^n|^2\right] \\
&\le 4\Bigg(\mathbb{E}\left[|\xi|^2 + |Y_t^n|^2\right] + \frac{1}{\beta}\mathbb{E}\left[\int_0^T e^{\beta A_s}\left|\frac{g(s)}{\alpha_s}\right|^2 d\langle M\rangle_s\right] \\
&\qquad + \mathbb{E}\left[\int_t^T |Z_s^n|^2 d\langle M\rangle_s\right] + \mathbb{E}\left[\int_t^T d[N^n]_s\right]\Bigg).
\end{aligned} \tag{3.13}$$

Choosing $\epsilon > 4\vee 4/\beta$ and evolving inequalities (3.13), (3.12) to (3.11), we deduce that

$$\mathbb{E}\left[\int_0^T e^{\beta A_s}|Y_s^n\alpha_s|^2 d\langle M\rangle_s\right] + \mathbb{E}\left[\int_0^T e^{\beta A_s}|Y_s^n|^2 ds\right] + \mathbb{E}\left[\int_0^T e^{\beta A_s}|Z_s^n|^2 d\langle M\rangle_s\right]$$

$$
\begin{aligned}
&+\mathbb{E}\left[\int_0^T e^{\beta A_s} d[N^n]_s\right]+\mathbb{E}\left[\left|K_T^n\right|^2\right]\\
\le{}& C_\beta\left\{\mathbb{E}\left[e^{\beta A_T}|\xi|^2\right]+\mathbb{E}\left[\int_0^T e^{\beta A_s}\left|\frac{g(s)}{\alpha_s}\right|^2 d\langle M\rangle_s\right]+\mathbb{E}\left[\sup_{0\le t\le T} e^{2\beta A_t}(L_t^+)^2\right]\right\}.
\end{aligned}
$$

From this, the formula (3.10), Gronwall's inequality and the Burkholder-Davis-Gundy's inequality, we deduce that for any $n\in\mathbb{N}$,

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T} e^{\beta A_t}\left|Y_t^n\right|^2\right]+\mathbb{E}\left[\int_0^T e^{\beta A_s}\alpha_s^2\left|Y_s^n\right|^2 d\langle M\rangle_s\right]+\mathbb{E}\left[\left|K_T^n\right|^2\right]\\
&\quad+\mathbb{E}\left[\int_0^T e^{\beta A_s}\left|Z_s^n\right|^2 d\langle M\rangle_s\right]+\mathbb{E}\left[\int_0^T e^{\beta A_s} d[N^n]_s\right]\\
&\le C_\beta\left\{\mathbb{E}\left[e^{\beta A_T}|\xi|^2\right]+\mathbb{E}\left[\int_0^T e^{\beta A_s}\left|\frac{g(s)}{\alpha_s}\right|^2 d\langle M\rangle_s\right]+\mathbb{E}\left[\sup_{0\le t\le T}\left|e^{\beta A_t}(L_t)^+\right|^2\right]\right\}.
\end{aligned}
$$

The proof of lemma is complete. □

Step 2: Weak convergence of the sequence $\{Y^n,Z^n,K^n,N^n\}_{n\in\mathbb{N}}$ to the solution (Y,Z,K,N) of the classical BSDE (2.1a) associated with (ξ,g) along a subsequence.

First, it is clear, that $f_{n+1}(s,y)\le f_n(s,y)$ for all $y\in\mathbb{R},\mathbb{P}(d\omega)\otimes dt$-a.e. and each $n\in\mathbb{N}$. Hence, it follows from the comparison theorem (see Appendix 4.2.2) that $Y^{n+1}\ge Y^n$ a.s. Therefore, there exits an $\mathscr{F}_t$-progressively measurable process $(Y_t)_{t\le T}$ such that $Y_t^n\nearrow Y_t, t\le T$, a.s. Fatou's lemma combined with the uniform estimation (3.9), gives

$$
\mathbb{E}\left[\sup_{0\le t\le T} e^{\beta A_t}|Y_t|^2\right]\le\liminf_{n\to+\infty}\mathbb{E}\left[\sup_{0\le t\le T} e^{\beta A_t}\left|Y_t^n\right|^2\right]\le\mathfrak{C}_\beta. \tag{3.14}
$$

Now, from the Hilbert structure of $\mathscr{H}_\beta^2$, we can extract sub-sequence of $\{Z^n\}_{n\in\mathbb{N}}$, which weakly converge in the related spaces to some process $(Z_t)_{t\le T}\in\mathscr{H}_\beta^2$.

Our next topic is the convergence of the sequence of orthogonal martingales $\{N^n\}_{n\in\mathbb{N}}$. Thanks to (3.9), we can conclude that

$$
\sup_{n\in\mathbb{N}}\mathbb{E}\left[\sup_{0\le t\le T}\left(\int_0^t e^{\frac{\beta A_s}{2}} dN_s^n\right)^2\right]<\infty,
$$

which follows from the BDG inequality. Thus, there exists a square-integrable $\mathbb{F}$-martingale N such that the stochastic integrals $\int_0^\cdot e^{\beta A_s/2} dN_s^n$ converge weakly to $\int_0^{\cdot} e^{\beta A_s/2} dN_s$ in $\mathbb{L}^2(\Omega, d\mathbb{P})$. More precisely, the process N is defined as the RCLL version of the square-integrable $\mathbb{F}$-martingale $\mathbb{E}[\chi \mid \mathscr{F}_\cdot]$, where χ is an $\mathscr{F}_T$-measurable square-integrable random variable such that $N^n_\tau \to N_\tau$ weakly in $\mathbb{L}^2(\mathscr{F}_\tau)$ for every $\tau \in \mathscr{T}_0^T$. The existence of such a modification is due to the right-continuity of the filtration $\mathbb{F}$. From the fact that the set of orthogonal martingales is closed and stable [44, Lemma 1, p. 180], we deduce that N is also orthogonal to M. Thanks to the predictable representation property (see [39, Remark 2.1, p. 323]), for any $\tau \in \mathscr{T}_0^T$, the following weak convergence holds in $\mathbb{L}^2(\mathscr{F}_\tau)$:

$$\int_0^\tau Z_s^n dM_s \rightharpoonup \int_0^\tau Z_s dM_s, \quad \int_0^\tau dN_s^n \rightharpoonup \int_0^\tau dN_s \quad \text{when} \quad n \to +\infty.$$

Since,

$$K_\tau^n = Y_\tau^n - Y_0^n - \int_0^\tau g(s) d\langle M \rangle_s + \int_0^\tau Z_s^n dM_s + \int_0^\tau dN_s^n,$$

by passing to the weak limit term-by-term in $\mathbb{L}^2(\mathscr{F}_\tau)$, we get

$$K_\tau^n \rightharpoonup K_\tau = Y_\tau - Y_0 - \int_0^\tau g(s) d\langle M \rangle_s + \int_0^\tau Z_s dM_s + \int_0^\tau dN_s.$$

Since the processes $(Y_t^n)_{t \le T}$ and $(L_t)_{t \le T}$ are two optional processes, hence an application of Fubini's theorem for transition measures shows that $(K_t^n)_{t \le T}$ is $\mathscr{F}_t$-adapted and then $(K_t^n)_{t \le T}$ is $\mathscr{F}_t$-predictable. Additional, the limit process K remains an increasing $\mathscr{F}_t$-predictable process since the sequence $(K_t^n)_{t \le T}$ is increasing predictable (K is equal to its dual predictable projection) with $K_0^n = 0$ and $\mathbb{E}[|K_T|^2] < \infty$. Furthermore, the processes Y and K are RCLL processes (see [43, Lemma 2.2]) and then we get the following decomposition of Y:

$$Y_t = \xi + \int_t^T g(s) d\langle M \rangle_s + (K_T - K_t) - \int_t^T Z_s dM_s - \int_t^T dN_s, \quad t \in [0,T]. \tag{3.15}$$

Step 3: Strong convergence of $\{Y^n, Z^n, K^n, N^n\}_{n \in \mathbb{N}}$ to (Y, Z, K, N) in $\mathbb{L}^2(\Omega, d\mathbb{P})$ along a subsequence.

We will now face new challenges because the sequence $(Y^n)_{n \in \mathbb{N}}$ and its weak limit Y are just RCLL process. Their jumps are caused by the martingale M, the added orthogonal martingale terms as well as the predictable process K (whereas K^n is continuous). More precisely, note that the processes Y^n defined by (3.8) has only totally inaccessible jumps, while the process Y solution of the generalized BSDE (3.15) has general jumps i.e. both the inaccessible and the predictable. The latest of jumps are carried by K, which we would like to control. As in [45], it should be noted that we may demonstrate that the contribution of the jumps of $(K_t)_{t\leq T}$ is mostly controlled on each interval $]\sigma_k,\tau_k]$ with sufficiently small total length, where σ_k ,τ_k are two $\mathscr{F}_t$-predictable stopping times. In other words, for any fixed λ and δ in $[0,1[$, there exists a finite sequence $(\sigma_k,\tau_k)_{k\in 0,\dots,N}$ such that

$$\bigcup_{k=0}^{N}]\sigma_k,\tau_k]=[0,T],\quad]\sigma_i,\tau_i]\cap]\sigma_j,\tau_j]=\emptyset,\quad \forall i\neq j,$$

and

$$\mathbb{E}\left[\sum_{k=0}^{N}(\tau_k-\sigma_k)\right]\geq T-\lambda,\quad \sum_{k=0}^{N}\mathbb{E}\left[\sum_{\sigma_k<s\leq\tau_k}\left|\Delta K_s^d\right|^2\right]\leq\frac{\lambda\delta}{2}.$$

We are now looking into the convergence of the martingale parts. By applying the same Itô's formula to the semimartingale $|Y_t^n-Y_t|^2$ given by (3.8)-(3.15) on each sub interval $0\leq\sigma<\tau\leq T$, where σ, τ are two $\mathbb{F}$-predictable times, we obtain

$$\begin{aligned}
&\left|Y_\sigma^n-Y_\sigma\right|^2+\int_\sigma^\tau\left(Z_s^n-Z_s\right)d\langle M,N^n-N\rangle_s^c\\
&\quad+\int_\sigma^\tau|Z_s-Z_s'|^2d\langle M\rangle_s+\int_\sigma^\tau d\langle N^n-N\rangle_s^c+\sum_{\sigma<s\leq\tau}e^{\beta A_s}\left(\Delta\left(Y_s^n-Y_s\right)\right)^2\\
&=\left|Y_\tau^n-Y_\tau\right|^2+2\int_\sigma^\tau\left(Y_s^n-Y_s\right)dK_s^n-2\int_\sigma^\tau\left(Y_{s-}^n-Y_{s-}\right)dK_s\\
&\quad+2\int_\sigma^\tau\left(Y_{s-}^n-Y_{s-}\right)\left(Z_s^n-Z_s\right)dM_s+2\int_\sigma^\tau(Y_{s-}-Y_{s-}')d(N^n-N)_s. \qquad (3.16)
\end{aligned}$$

Note that,

$$\int_\sigma^\tau\left(Y_{s-}^n-Y_{s-}\right)dK_s=\int_\sigma^\tau\Delta\left(Y_s-Y_s^n\right)dK_s+\int_\sigma^\tau\left(Y_s^n-Y_s\right)dK_s,$$

and similarly to (2.2), we have

$$\begin{aligned}
\int_\sigma^\tau\Delta\left(Y_s-Y_s^n\right)dK_s=&\int_\sigma^\tau\left(Z_s-Z_s^n\right)\Delta M_sdK_s+\int_\sigma^\tau\Delta N_sdK_s\\
&-\int_\sigma^\tau\Delta N_s^ndK_s-\int_\sigma^\tau\Delta K_sdK_s.
\end{aligned}$$

From this and [45, Lemma A.1], we get

$$\mathbb{E}\left[\int_\sigma^\tau\Delta\left(Y_s-Y_s^n\right)dK_s\right]=\mathbb{E}\left[\int_\sigma^\tau\Delta\left(Y_s-Y_s^n\right)dK_s^d\right]$$

$$
\begin{aligned}
&= \mathbb{E}\Big[\sum_{\sigma<s\leq\tau}(Z_s - Z_s^n)\Delta M_s \Delta K_s^d + \sum_{\sigma<s\leq\tau}\Delta N_s \Delta K_s^d \\
&\qquad - \sum_{\sigma<s\leq\tau}\Delta N_s^n \Delta K_s^d - \sum_{\sigma<s\leq\tau}(\Delta K_s^d)^2\Big] \\
&= -\mathbb{E}\Big[\sum_{\sigma<s\leq\tau}(\Delta K_s^d)^2\Big].
\end{aligned}
$$

After plugging this into (3.16) with (2.2), (2.4), (2.5) and taking into account both side's expectations, we get

$$
\begin{aligned}
&\mathbb{E}\left[|Y_\sigma^n - Y_\sigma|^2\right] + \mathbb{E}\left[\int_\sigma^\tau |Z_s^n - Z_s|^2 d\langle M\rangle_s\right] + \mathbb{E}\left[\int_\sigma^\tau d[N^n - N]_s\right] \\
&\leq \mathbb{E}\left[|Y_\tau^n - Y_\tau|^2\right] + 2\mathbb{E}\left[\int_\sigma^\tau (Y_s^n - Y_s) dK_s^n\right] + \mathbb{E}\Big[\sum_{\sigma<s\leq\tau}(\Delta K_s^d)^2\Big]. \qquad (3.17)
\end{aligned}
$$

Now, for each $\sigma=\sigma_k$ and $\tau=\tau_k$, if we use estimate (3.17) it follows that

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{k=0}^N \int_{\sigma_k}^{\tau_k} |Z_s^n - Z_s|^2 d\langle M\rangle_s\right] + \mathbb{E}\left[\sum_{k=0}^N \int_{\sigma_k}^{\tau_k} e^{\beta A_s} d[N^n - N]_s\right] \qquad (3.18) \\
&\leq \mathbb{E}\left[\sum_{k=0}^N |Y_{\tau_k}^n - Y_{\tau_k}|^2\right] + 2\mathbb{E}\left[\sum_{k=0}^N \int_{\sigma_k}^{\tau_k} |Y_s^n - Y_s| dK_s^n\right] + \mathbb{E}\left[\sum_{k=0}^N \sum_{\sigma_k<s\leq\tau_k}(\Delta K_s^d)^2\right].
\end{aligned}
$$

At the current level, we want to show that $\int_0^\cdot Y_s^n ds \nearrow \int_0^\cdot Y_s ds$ $\mathbb{P}$-a.s. First, note that, the two stochastic integral $\int_0^\cdot (Y_s^n - Y_s) ds$ is indistinguishable from the Lebesgue-Stieltjes integral, computed path-by-path. Hence, the results of classical Dominate convergence theorem holds also true. The sequence $(Y^n)_{n\in\mathbb{N}}$ converges to Y a.s. and $Y^n \leq Y$ a.s., moreover, from (3.14), we have

$$
\mathbb{E}\left[\int_0^T |Y_s|^2 ds\right] \leq T\mathbb{E}\left[\sup_{0\leq t\leq T} e^{\beta A_s}|Y_s|^2\right] < \infty.
$$

Thus, by applying the dominated convergence theorem for stochastic integrals, we deduce that $\int_0^\cdot Y_s^n ds$ converges to $\int_0^\cdot Y_s ds$ in UCP. Then, we can deduce that $\int_0^\cdot Y_s^n ds$ converges to $\int_0^\cdot Y_s ds$ $\mathbb{P}$-a.s.

From (3.9), we have

$$
\mathbb{E}\left[\int_0^T |Y_s^n|^2 ds\right] \leq T\mathbb{E}\left[\sup_{0\leq t\leq T} e^{\beta A_s}|Y_s^n|^2\right] \leq T\mathfrak{C}_\beta,
$$

where the constant $\mathfrak{C}_\beta$ is independent of n. Henceforth, applying once more the dominated convergence theorem, we deduce that

$$\lim_{n\to+\infty}\mathbb{E}\left[\int_0^T \left|Y_s^n - Y_s\right|^2 ds\right] = 0.$$

Using Hölder's inequality for stochastic integrals, Cauchy-Schwarz inequality for random variables, inequalities (3.9), (3.14) and the convergence property above with $(a-b)^+ \leq a^+ + |b|$, we deduce that

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \left|Y_s^n - Y_s\right| dK_s^n\right]\\
&\leq \left(\mathbb{E}\left[\int_0^T \left|Y_s^n - Y_s\right|^2 ds\right]\right)^{\frac{1}{2}} \left(\mathbb{E}\left[\int_0^T \left((L_s - Y_s^n)^+\right)^2 ds\right]\right)^{\frac{1}{2}}\\
&\leq \sqrt{2}T\left(\mathbb{E}\left[\int_0^T \left|Y_s^n - Y_s\right|^2 ds\right]\right)^{\frac{1}{2}} \left(\mathbb{E}\left[\sup_{0\leq s\leq T}\left|e^{\beta A_s}(L_s)^+\right|^2\right] + \mathbb{E}\left[\sup_{0\leq s\leq T} e^{\beta A_s}\left|Y_s^n\right|^2\right]\right)^{\frac{1}{2}}\\
&\leq \sqrt{2}T\mathfrak{C}_\beta\left(\mathbb{E}\left[\int_0^T \left|Y_s^n - Y_s\right|^2\right]\right)^{\frac{1}{2}} \;\rightarrow\; 0 \quad \text{as} \quad n \;\rightarrow\; +\infty. \qquad (3.19)
\end{aligned}$$

Going back to (3.18), taking the convergence results (3.19) and the dominate convergence theorem into account, we obtain

$$\begin{aligned}
&\limsup_{n\to+\infty}\left\{\mathbb{E}\left[\sum_{k=0}^N \int_{\sigma_k}^{\tau_k} \left|Z_s^n - Z_s\right|^2 d\langle M\rangle_s\right] + \mathbb{E}\left[\sum_{k=0}^N \int_{\sigma_k}^{\tau_k} d[N^n - N]_s\right]\right\}\\
&\leq \mathbb{E}\left[\sum_{k=0}^N \sum_{\sigma_k < s \leq \tau_k} \left(\Delta K_s^d\right)^2\right] \leq \frac{\lambda\delta}{2}.
\end{aligned}$$

Hence, there exists $N(\lambda,\delta)\in\mathbb{N}$ such that for any $n \geq N(\lambda,\delta)$, we obtain

$$\mathbb{E}\left[\int_0^T \left|Z_s^n - Z_s\right|^2 d\langle M\rangle_s\right] + \mathbb{E}\left[\int_0^T d[N^n - N]_s\right] \leq \frac{\lambda\delta}{2}.$$

Using Chebyshev's inequality, we can prove that

$$d\langle M\rangle(\cdot)\otimes d\mathbb{P}(\cdot)\left\{(\omega,s)\in\Omega\times[0,T] : \left|Z_s^n(\omega) - Z_s(\omega)\right|^2 \geq \delta\right\} \leq \frac{\lambda}{2},$$

where $d\mathbb{P}\otimes d\langle M\rangle$ is the positive measure on $(\Omega\times[0,T], \mathscr{F}\otimes\mathscr{B}([0,T]))$ defined for any $\mathscr{V}\in\mathscr{F}\otimes\mathscr{B}([0,T])$ by

$$d\mathbb{P}\otimes d\langle M\rangle(\mathscr{V}) = \mathbb{E}\left[\int_0^T \mathbb{1}_{\mathscr{V}}(\omega,s) d\langle M\rangle_s\right],$$

and

$$\mathbb{P}\left\{\omega\in\Omega : \int_0^T d[N^n-N]_s(\omega)\geq\delta\right\}\leq\frac{\lambda}{2}.$$

Thus, the sequence $(Z^n)_{n\in\mathbb{N}}$ (respectively $(\int_0^T d[N^n-N]_s)_{n\in\mathbb{N}}$) converges in measure (respectively in probability) to Z (respectively zero). In one hand, since $(Z^n)_{n\in\mathbb{N}}$ is uniformly bounded in $\mathscr{H}^2_\beta$ which is obviously a subspace of $\mathscr{H}^2$, then for any $p\in[0,2[$, we have

$$\lim_{n\to+\infty}\mathbb{E}\left[\int_0^T\left|Z_s^n-Z_s\right|^p d\langle M\rangle_s\right]=0.$$

On the other hand, we can extract a sub-sequence $(\int_0^T d[N^{n_k}-N]_s)_{k\in\mathbb{N}}$ that converges almost surely to 0. Note the $N^{n_k}-N$ is square integrable $\mathbb{F}$-martingale [44, Theorem 48, p. 37]. Using Cauchy-Schwarz and the BDG inequality, we can deduce that there exists a universal constant c such that the following inequality holds:

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|N_s^{n_k}-N_s\right|\right]\leq c\mathbb{E}\left[\int_0^T d\left[N^{n_k}-N\right]_s\right]+c\mathbb{E}\left[\left(N_0^{n_k}-N_0\right)^2\right].$$

By construction, we know that $N_\tau^{n_k}\rightharpoonup N_\tau$ in $\mathbb{L}^2(\mathscr{F}_\tau)$ as $k\to+\infty$. Additionally, considering the fact that $\mathscr{F}_0$ is the trivial σ-field, we can conclude that $N_0^k\to N_0$ in $\mathbb{R}$. Consequently, it follows that $\mathbb{E}[(N_0^{n_k}-N_0)^2]\to 0$ as $k\to+\infty$. Therefore, we have

$$\lim_{k\to+\infty}\mathbb{E}\left[\sup_{0\leq t\leq T}\left|N_s^{n_k}-N_s\right|\right]=0.$$

Step 4: Skorokhod condition.

Now, let us show the Skorokhod condition and that $Y_t\geq L_t$, $t\in[0,T]$. To that end, we present the following family of random variables indexed on $\mathscr{F}_t$ stopping times:

$$\begin{aligned}\bar{Y}(\sigma):&=\operatorname*{ess\,sup}_{\tau\in\mathscr{T}_\sigma^T}\mathbb{E}^{\mathscr{F}_\sigma}\left[\int_\sigma^\tau g(s)d\langle M\rangle_s+L_\tau\mathbb{1}_{\{\tau<T\}}+\xi\mathbb{1}_{\{\tau=T\}}\right]\\&=\operatorname*{ess\,sup}_{\tau\in\mathscr{T}_\sigma^T}\mathbb{E}^{\mathscr{F}_\sigma}\left[\int_0^\tau g(s)d\langle M\rangle_s+L_\tau\mathbb{1}_{\{\tau<T\}}+\xi\mathbb{1}_{\{\tau=T\}}\right]-G_\sigma\\&=:J(\sigma)-G_\sigma,\end{aligned}\tag{3.20}$$

where $G_\cdot:=\int_0^\cdot g(s)d\langle M\rangle_s$ and the process $(Y_t)_{t\leq T}$ is the solution of BSDE (3.15) constructed above. The latest equality is due to the fact that $G_\sigma\in\mathscr{F}_\sigma$ for every

$\sigma \in \mathscr{T}_0^T$. Indeed, it is easy to show that $(G_t)_{t\leq T}$ is an $\mathscr{F}_t$-predictable process and then the result is obtained directly from [30, Proposition 2.4]. Furthermore, following Dellacherie-Lenglart's terminology [9] it is clear that the family $\{J(\sigma),\sigma\in\mathscr{T}_0^T\}$ is a $\mathscr{T}_0^T$-supermartingale and from the fundamental aggregation theorems for stochastic systems (see Dellacherie and Lenglart [9] or [11, Chapter II] for more explicit details), there exist an RCLL $\mathscr{F}_t$-optional process $(J_t^\star)_{t\leq T}$ such that $J(\sigma)=J_\sigma^\star$ $\mathbb{P}$-a.s. for each σ in $\mathscr{T}_0^T$. Then, the $\mathscr{T}_0^T$-system $\bar{Y}$ can be aggregated by an RCLL $\mathscr{F}_t$-optional process $(\bar{Y}_t)_{t\leq T}$ such that $\bar{Y}_t := J_t^\star - G_t$ and $\bar{Y}(\sigma)=\bar{Y}_\sigma$. Now, we can deal with the aggregation version of the system (3.20) defined by

$$\bar{Y}_t = \underset{\tau\in\mathscr{T}_t^T}{\operatorname{esssup}} \mathbb{E}^{\mathscr{F}_t}\left[\int_t^\tau g(s)d\langle M\rangle_s + L_\tau \mathbb{1}_{\{\tau<T\}} + \xi\mathbb{1}_{\{\tau=T\}}\right] = J_t^\star - G_t. \tag{3.21}$$

Let $\eta := (\eta_t)_{t\leq T}, \eta^+ := (\eta_t^+)_{t\leq T}$ and $M^\star := (M_t^\star)_{t\leq T}$ be the processes defined as follows:

$$\begin{aligned} \eta_t &= \int_0^t g(s)d\langle M\rangle_s + L_t\mathbb{1}_{\{t<T\}} + \xi\mathbb{1}_{\{t=T\}}, \\ M_t^\star &= \mathbb{E}\left[\xi + \int_0^T g(s)d\langle M\rangle_s \,|\, \mathscr{F}_t\right], \\ \eta_t^+ &= \int_0^t g(s)d\langle M\rangle_s + L_t^+\mathbb{1}_{\{t<T\}} + \xi\mathbb{1}_{\{t=T\}}. \end{aligned}$$

It is clear that

$$M_t^\star \leq J_t^\star = \underset{\tau\in\mathscr{T}_0^T}{\operatorname{esssup}} \mathbb{E}[\eta_\tau \,|\, \mathscr{F}_t] \leq \underset{\tau\in\mathscr{T}_0^T}{\operatorname{esssup}} \mathbb{E}[\eta_\tau^+ \,|\, \mathscr{F}_t] =: J_t^{\star\star} \quad \mathbb{P}\text{-a.s.} \tag{3.22}$$

From (2.3) we have, for any $\tau\in\mathscr{T}_0^T$,

$$\mathbb{E}\left[\left(\xi + \int_0^T g(s)d\langle M\rangle_s\right)^2\right] \leq 3\left(\mathbb{E}\left[e^{\beta A_T}|\xi|^2\right] + \frac{1}{\beta}\mathbb{E}\left[\int_0^T e^{\beta A_s}\left|\frac{g(s)}{\alpha_s}\right|^2 d\langle M\rangle_s\right]\right).$$

Thus, $(M_t^\star)_{t\leq T}$ is a square-integrable $\mathbb{F}$-martingale and

$$\sup_{0\leq t\leq T}\left[e^{\beta A_t}\eta^+\right] \in \mathbb{L}^2(\mathscr{F}_T).$$

Using (3.22) with Doob's maximal inequality ([30, Theorem 1.43, p. 11]), we deduce that

$$\mathbb{E}\left[\sup_{0\leq s\leq T}|J_s^\star|^2\right] \leq \max\left(\mathbb{E}\left[\sup_{0\leq s\leq T} e^{\beta A_s}|J_s^{\star\star}|^2\right], \mathbb{E}\left[\sup_{0\leq s\leq T}|M_s^\star|^2\right]\right) < \infty.$$

Hence, the Snell envelope $(J_t^\star)_{t\le T}$ of the process η satisfies $\mathbb{E}[\sup_{0\le t\le T}|J_t^\star|^2]<\infty$ and then it is of class $\mathscr{D}([0,T])$. Equivalently, the process $J_t^\star=\bar{Y}_t+G_t$ is an $\mathbb{F}$-supermartingale of class $\mathscr{D}([0,T])$, so it has the following Doob-Meyer decomposition [44, Theorem 8, p. 111]:

$$J_t^\star=J_0^\star+\bar{M}_t-\bar{K}_t,\quad 0\le t\le T, \tag{3.23}$$

where $\bar{K}$ is a non-decreasing, RCLL, $\mathscr{F}_t$-predictable process such that $\bar{K}_0=0$ and $\mathbb{E}[|\bar{K}_T|^2]<\infty$ and $\bar{M}$ is a square-integrable $\mathbb{F}$-martingale. Through the martingale representation property, there exists a unique pair of processes $(\bar{Z},\bar{N})\in\mathscr{H}^2\times\mathscr{M}^2$ such that $\bar{M}_t=\int_0^t\bar{Z}_sdM_s+\int_0^t d\bar{N}_s$. Therefore, from (3.21) and (3.23), we conclude that the process $(\bar{Y}_t)_{0\le t\le T}$ satisfies the following BSDE:

$$\bar{Y}_t=\xi+\int_t^T g(s)d\langle M\rangle_s+(\bar{K}_T-\bar{K}_t)-\int_t^T\bar{Z}_sdM_s-\int_t^T d\bar{N}_s. \tag{3.24}$$

The quadruplet $(Y_t,Z_t,N_t,K_t)_{t\le T}$ is uniquely determined by construction, thanks to the strong convergence property satisfied by a subsequence $\{(Y^n,Z^n,K^n,N^n)\}_{n\in\mathbb{N}}$ as shown in Step 3. Thus, BSDEs (3.15) and (3.24) has the same solution, then, we deduce that $(Y,Z,K,N)=(\bar{Y},\bar{Z},\bar{K},\bar{N})$, and the limited process $(Y_t)_{t\le T}$ in the weak sense of the increasing penalization schemes has the following expression:

$$Y_t=\operatorname*{esssup}_{\tau\in\mathscr{T}_t^T}\mathbb{E}^{\mathscr{F}_t}\left[\int_t^\tau g(s)d\langle M\rangle_s+L_\tau\mathbb{1}_{\{\tau<T\}}+\xi\mathbb{1}_{\{\tau=T\}}\right]. \tag{3.25}$$

From the very definition of the essential sup, we get for $\tau=t$,

$$Y_t\ge L_t\mathbb{1}_{\{t<T\}}+\xi\mathbb{1}_{\{t=T\}}\ge L_t\mathbb{1}_{\{t<T\}}+L_T\mathbb{1}_{\{t=T\}}=L_t\quad \mathbb{P}\text{-a.s.}$$

It remains to show the minimality condition $\int_0^T(Y_{t-}-L_{t-})dK_t=0$. For this, note that, from (3.25), the state process $(Y_t)_{t\le T}$ can be written as

$$Y_t=\mathfrak{R}_t(\eta^\star)+\mathbb{E}\left[\xi+\int_t^T g(s)d\langle M\rangle_s\,|\,\mathscr{F}_t\right],$$

where $\mathfrak{R}_\cdot(\eta^\star)$ is the Snell envelope of the reward process $(\eta_t^\star)_{0\le t\le T}$ defined by

$$\eta_t^\star:=\eta_t-\mathbb{E}\left[\xi+\int_0^T g(s)d\langle M\rangle_s\,|\,\mathscr{F}_t\right].$$

Using the Doob-Meyer decomposition to the $\mathbb{F}$-supermartingale $\mathfrak{R}_\cdot(\eta^\star)$ one more, the martingale representation property, BSDE (3.15), and the fact that each $\mathscr{F}_t$-

predictable martingale of finite variation is constant [30, Corollary 3.16, p. 32], we conclude that

$$\mathfrak{R}_t(\eta^\star)=Y_t-\mathbb{E}\left[\xi+\int_t^T g(s)d\langle M\rangle_s\,|\,\mathscr{F}_t\right]=\int_0^t Z_s dM_s+\int_0^t dN_s-K_t.$$

Recall that the process $(K_t)_{t\leq T}$ admits the unique decomposition $K_t=K_t^d+K_t^c$, where $(K_t^c)_{t\leq T}$ is the continuous part of $(K_t)_{t\leq T}$ and (K_t^d) is its purely discontinuous part.

Now, making use of the results obtained in the optimal stopping theory (see [28, Propositions (A2)-(A3)] or [11] for more details on the properties of the Snell envelope), we obtain $\{\Delta K^d>0\}\subset\{\mathfrak{R}_-(\eta^\star)=\eta^\star_-\}$ and

$$\Delta K_t^d=\left(\eta^\star_{t-}-\mathfrak{R}_t(\eta^\star)\right)^+\mathbb{1}_{\{\eta^\star_{t-}=\mathfrak{R}_{t-}(\eta^\star)\}}.$$

Henceforth,

$$\int_0^T (Y_{t-}-L_{t-})dK_t^d=\sum_{0<t\leq T}(Y_{t-}-L_{t-})\Delta K_t^d=\sum_{0<t\leq T}\left(\mathfrak{R}_{t-}(\eta^\star)-\eta^\star_{t-}\right)\Delta K_t^d=0.$$

It is easy to see that $\mathfrak{R}(\eta^\star+K^d)=\mathfrak{R}(\eta^\star)+K^d$ and $\mathfrak{R}(\eta^\star)+K^d=M-K^c$ is a regular $\mathbb{F}$-supermartingale, i.e. ${}^p(\mathfrak{R}(\eta^\star)+K^d)=\mathfrak{R}_-(\eta^\star)+K^d_-$, where ${}^p(\mathfrak{R}(\eta^\star)+K^d)$ stands for the $\mathbb{F}$-predictable projection of the process $\mathfrak{R}(\eta^\star)+K^d$ (for more information on process projection, see [29, Section 1, Chapter V, p. 135]), then the stopping time $\nu_t=\inf\{s\geq t:K_t^c>K_s^c\}\wedge T$ is optimal after t and then

$$\begin{aligned}&\int_t^{\nu_t}\left(\mathfrak{R}_s(\eta^\star+K^d)-\left(\mathfrak{R}_s(\eta^\star)+K_s^d\right)\right)dK_s^c\\&=\int_t^{\nu_t}\left(\mathfrak{R}_s(\eta^\star)-\eta^\star_s\right)dK_s^c=\int_t^{\nu_t}(Y_s-L_s)dK_s^c=0\end{aligned}$$

for any arbitrary $t\in[0,T]$. Thus, $\int_0^T(Y_s-L_s)dK_s^c=0$.

The constructed process $(Y_t,Z_t,K_t,N_t)_{t\leq T}$ is then the unique solution of the reflected BSDE associated with (ξ,g,L). Furthermore, as we pass to the limit on a subsequence using Fatou's lemma, it becomes evident from (3.9) that the quadruplet $(Y_t,Z_t,K_t,N_t)_{t\leq T}$ satisfies

$$\begin{aligned}&\mathbb{E}\left[\sup_{0\leq t\leq T}e^{\beta A_t}|Y_t|^2\right]+\mathbb{E}\left[\int_0^T e^{\beta A_s}\alpha_s^2|Y_s|^2d\langle M\rangle_s\right]\\&\quad+\mathbb{E}\left[\int_0^T e^{\beta A_s}|Z_s|^2d\langle M\rangle_s\right]+\mathbb{E}\left[|K_T|^2\right]+\mathbb{E}\left[\int_0^T e^{\beta A_s}d[N]_s\right]\end{aligned}$$

$$\leq C_\beta \left\{ \mathbb{E}\left[e^{\beta A_T}|\xi|^2\right] + \mathbb{E}\left[\int_0^T e^{\beta A_s}\left|\frac{g(s)}{\alpha_s}\right|^2 d\langle M\rangle_s\right] + \mathbb{E}\left[\sup_{0\leq t\leq T}|e^{\beta A_t}(L_t)^+|^2\right]\right\}.$$

The proof of Theorem 3.2 is complete. □

Part 2: General case.

At this stage, we are ready to present the main result of this section. Specifically, we will establish the existence and uniqueness of a solution for the RBSDE (2.1) by utilizing the results obtained in previous sections. This will be achieved by identifying a fixed point of the contraction mapping defined by the function Ψ in the following manner.

Let $\mathfrak{B}^2_\beta := \mathscr{S}^{2,\alpha}_\beta \times \mathscr{H}^2_\beta \times \mathscr{M}^2_\beta$ endowed with the norm

$$\|(Y,Z,N)\|_\beta = \left(\mathbb{E}\left[\int_0^T e^{\beta A_s}\left\{|\alpha_s Y_s|^2 + |Z_s|^2\right\} d\langle M\rangle_s + d[N]_s\right]\right)^{\frac{1}{2}}.$$

Let Ψ be the map from $\mathfrak{B}^2_\beta$ into itself which with (X,W,O) associates $(Y,Z,N) = \Psi(X,W,O)$,where (Y,Z,N) is the solution of the doubly reflected BSDE associated with $(\xi, f(t,X_t,W_t), L)$. Let (X',W',O') be another triple of $\mathfrak{B}^2_\beta$ and $(Y',Z',N') = \Psi(X',W',O')$. Set $\bar{\mathfrak{S}} = \mathfrak{S} - \mathfrak{S}'$, for $\mathfrak{S} = Y,Z,K,N,X,W$ and O.

Using Itô's formula, we obtain for any $t \leq T$,

$$\begin{aligned}
&\beta \mathbb{E}\left[\int_0^T e^{\beta A_s}|\alpha_s \bar{Y}_s|^2 d\langle M\rangle_s\right] + \mathbb{E}\left[\int_0^T e^{\beta A_s}|\bar{Z}_s|^2 d\langle M\rangle_s\right] + \mathbb{E}\left[\int_0^T e^{\beta A_s} d[\bar{N}_s]_s\right]\\
&\leq 2\mathbb{E}\left[\int_t^T e^{\beta A_s}\bar{Y}_s\big(f(s,X_s,W_s) - f(s,X'_s,W'_s)\big) d\langle M\rangle_s\right] + 2\mathbb{E}\left[\int_t^T e^{\beta A_s}\bar{Y}_{s-} d\bar{K}_s\right]\\
&\leq |\beta - 1|\mathbb{E}\left[\int_0^T e^{\beta A_s}|\alpha_s \bar{Y}_s|^2 d\langle M\rangle_s\right]\\
&\quad + \frac{3}{|\beta-1|}\mathbb{E}\left[\int_0^T e^{\beta A_s}\big(\left\{|\alpha_s \bar{X}_s|^2 + |\bar{W}_s|^2\right\} d\langle M\rangle_s + d[\bar{O}]_s\big)\right].
\end{aligned}$$

Here, we have relied on the hypotheses **(H2)**(ii) and the Skorokhod condition, which implies

$$(Y_{s-} - Y'_{s-})(dK_s - dK'_s) \leq 0.$$

Choosing $\beta > 4$, then Ψ is a strict contraction mapping on the Banach space $\mathfrak{B}^2_\beta$, henceforth, there exists a triple of processes $(Y_t, Z_t, N_t)_{t\leq T}$ that is a fixed point to such that $\Psi(Y,Z,N)=(Y,Z,N)$, which, with K, is the unique solution of the doubly reflected BSDE associated with (ξ,f,L) and the sate $(Y_t)_{t\leq T}$ can be expressed as

$$Y_t=\underset{\tau\in\mathscr{T}_t^T}{\operatorname{esssup}}\mathbb{E}^{\mathscr{F}_t}\left[\int_t^\tau g(s,Y_s,Z_s)d\langle M\rangle_s+L_\tau\mathbb{1}_{\{\tau<T\}}+\xi\mathbb{1}_{\{\tau=T\}}\right].$$

4 Application to computing American option prices in financial markets driven by Azéma's martingale

4.1 The Azéma martingale

Here we assume that the filtration $\mathbb{G}=(\mathscr{G}_t)_{t\leq T}$ is the one generated by the martingale $(M_t)_{t\leq T}$ made right-continuous and complete, characterized by the so-called structure equation

$$d[M,M]_t=dt-M_{t-}dM_t. \tag{4.1}$$

The existence and (weak) uniqueness of a solution for the structure equation (4.1) are guaranteed by the seminal works of Meyer [38] and Emery [21] respectively.

The unique solution $(M_t)_{t\leq T}$ of the Eq. (4.1) is called the Azéma martingale. Recall that:

- The continuous and purely discontinuous parts of M are given by

$$dM_t^c=\mathbb{1}_{\{M_{t-}=0\}}dM_t,$$
$$dM_t^d=\mathbb{1}_{\{M_{t-}\neq0\}}dM_t.$$

- When M jumps at some stopping time $\tau\in\mathscr{T}_0^T$, then
  - τ is totally inaccessible,
  - the jump size ΔM_τ is equal to $-M_{\tau-}$,
  - after the jump, we get $M_\tau=0$.

- $(M_t)_{t\leq T}$ has the chaotic representation property (see [21, Proposition 6(ii), p. 80]), in particular, it has the martingale's predictable representation property (the reader is referred to [40] for more details on the chaos representation and the references within it).

- Also, note that $\langle M,M\rangle_t=t$ is a consequence of the structure equation, since $([M,M]_t-t)_{t\leq T}$ is a true martingale. This is due to the boundedness of the martingale $(M_t)_{t\leq T}$, as shown in [21, p. 83].

4.2 The market model

4.2.1 Problem definition and a real-life example

This section pertains to the evaluation of an American contingent claim that is traded between two financial counterparts: a regular option seller, such as a company that employs investment and risk management strategies. The term "regular" implies that this investor has access to public information that is available through the market. The other counterpart is an insider, who is the option buyer and an individual with access to non-public information. In this context, the insider is the holder in the given financial market that is driven by Azéma's martingale $(M_t)_{t\leq T}$, as defined above.

It is important to note that, in financial jargon and default risk modeling, an insider is an individual who has access to confidential information about a company, which may include financial results, business plans, or upcoming events that could have an impact on the company's stock price.

For illustration, let us say an insider at a firm discovers that the company is set to reveal a substantial advancement in a new product development that is predicted to greatly boost the stock price of the company. Before the announcement, the insider chooses to purchase American call options on the company's stock in an effort to profit from the expected increase in the stock price.

This type of situation has been encountered many times in real economic life. One well-known example of insider trading involving American options occurred in the case of United States against Todd Newman. More precisely, in 2011, two hedge fund analysts obtained private knowledge about impending earnings statements from insiders of technology companies. They exploited this information to trade American call options on the stocks of these companies before the earnings reports were made public, and they profited handsomely as a result.

The aim of this section is to utilize the one reflected BSDEs explored in Section 2 to calculate the fair value of the American option, which represents the initial payment made by the holder at time $t=0$.

4.2.2 Model formulation and solution approach

Market model. Let us now investigate the cost problem of an American option, which involves a seller (a company) and a buyer (an insider) in a financial market governed by the dynamics of the Azéma martingale (4.1). The evolution of the flow of information is determined by the filtration $\mathbb{G}=(\mathscr{G}_t)_{t\leq T}$ generated by M. In order to do so, we consider a financial market comprising two derivative securities – a non-risky asset (such as a bond) and a risky one (such as a stock) –

and suppose that the prices of these securities are determined by the following system of stochastic differential equations:

$$\begin{cases} dS^0_t = r_t S^0_t dt, & S^0_0 = 1, \\ dS_t = S_{t-} dM_t, & S_0 = 1, \end{cases} \tag{4.2}$$

where $(r_t)_{t\leq T}$ is a positive process representing the interest rate. On the other hand, at time $t=0$, the buyer of the option knows, in addition to the public information $\mathbb{G}$, an $\mathscr{F}$-measurable random variable X. The "natural" filtration known by the insider trader (buyer) is $\mathscr{Q}_t = \mathscr{G}_t \vee \sigma(X)$. Obviously, in our situation, the problem of valuing the American contingent claims cannot be solved on either $(\mathscr{G}_t)_{t\leq T}$ or $(\mathscr{Q}_t)_{t\leq T}$, but on the filtration that provides the total information, i.e. the one that contains $\mathbb{G}$ and the set of additional information $\sigma(X)$. In other words, to apply the standard results, we use the associated right and quasi-left-continuous filtration, denoted by $\mathbb{F} = (\mathscr{F}_t)_{t\leq T}$

$$\mathscr{F}_t = \bigcap_{\epsilon>0} \{\mathscr{G}_{t+\epsilon} \vee \sigma(X)\}, \quad t \in [0,T],$$

completed by all the $\mathbb{P}$-null sets of $\mathscr{F}$. This is known as the initial enlargement of the filtration $\mathbb{G}$ by the random variable X.

Remark 4.1.

- In general, the semimartingale property is not always stable in the enlarged filtration $\mathbb{F}$. More precisely, there is no theorem that guarantees that every $\mathbb{G}$-semimartingale is a $\mathbb{F}$-semimartingale. The latter property is known as the $(\mathscr{H}')$-assumption. For more details about this assumption, readers may consult [31,37,46] and the references therein.
- In our situation, the problem of valuing game contingent claims is well-defined. This is because the $(\mathbb{G},\mathbb{P})$-martingale $(M_t)_{t\leq T}$ remains an $(\mathbb{F},\mathbb{P})$-martingale, due to the martingale representation property which is relaxed in our case by introducing a new martingale orthogonal term in the definition of the solution. Therefore, we do not require any additional assumptions on the random variable X.
- Since the martingale $(M_t)_{t\leq T}$ satisfies the structural equation (4.1) path by path almost surely under $d\mathbb{P}$, we can deduce that M is also an Azéma martingale in the new filtration $\mathbb{F}$.
- Several authors, including Akdim [1], Akdim, Diakhaby and Ouknine [2], El Otmani [14] and Øksendal and Zhang [41], have explored the relationship between the problem of insider trading and the theory of BSDEs.

Completeness and free of arbitrage property of the market model (4.2)**.**

- Note that the riskless asset price $(S_t^0)_{t\leq T}$ satisfies a certain form of a linear Cauchy problem. To ensure the uniqueness of such a solution, we make an additional assumption of continuous paths for the process $(r_t)_{t\leq T}$, in addition to the positivity assumption. Under these conditions, the process $(S_t^0)_{t\leq T}$ can be expressed as

$$S_t^0 = e^{\int_0^t r_s ds}, \quad 0\leq t\leq T.$$

- Observe that the risky asset price $(S_t)_{t\leq T}$ has the following closed form (see [44, Theorem 37, p. 84]):

$$S_t = \exp\left(M_t - \frac{1}{2}[M,M]_t^c\right)\prod_{0<s\leq t}(1-M_{s-})e^{\Delta M_s},$$

and we have the following proposition.

Proposition 4.1. *The risky asset price $(S_t)_{t\leq T}$ is an $(\mathbb{F},\mathbb{P})$-martingale and satisfies $S_t>0$ for any $t\in[0,T]$.*

Proof. Note that the dynamics of the process $(S_t)_{t\leq T}$ can be written in the following integral form:

$$S_t = 1+\int_0^t S_{s-}dM_s, \quad t\in[0,T]. \tag{4.3}$$

Hence, as stated in the second point of Remark 4.1, we can conclude that $(S_t)_{t\leq T}$ is an $\mathbb{F}$-local martingale. Next, we define the following sequence of stopping times: $\tau_n = \inf\{t\geq 0: |S_t|\geq n\}\wedge T$. It is clear that the sequence $\{\tau_n\}_{n\in\mathbb{N}}$ is increasing and that $\tau_n \nearrow T$ a.s., since the trajectories of the process S are left-limited at each point in $[0,T]$. For (4.3) and Fubini's theorem, we can derive the inequality

$$\begin{aligned}\mathbb{E}\left[|S_t|^2\mathbb{1}_{\{t\leq\tau_n\}}\right] &\leq \mathbb{E}\left[|S_{t\wedge\tau_n}|^2\right] \leq 2\left(1+\mathbb{E}\left[\left|\int_0^{t\wedge\tau_n} S_{s-}dM_s\right|^2\right]\right)\\ &\leq 2\left(1+\mathbb{E}\left[\int_0^{t\wedge\tau_n}|S_{s-}|^2 ds\right]\right)\\ &= 2\left(1+\int_0^t \mathbb{E}\left[|S_s|^2\mathbb{1}_{\{s\leq\tau_n\}}\right]ds\right).\end{aligned}$$

Using Gronwall's inequality and Fatou's lemma, we obtain

$$\mathbb{E}\left[|S_t|^2\right] \leq \liminf_{n\to+\infty}\mathbb{E}\left[|S_t|^2\mathbb{1}_{\{t\leq\tau_n\}}\right] \leq 2e^{2T}, \quad \forall t\in[0,T].$$

Therefore, we can conclude that M is uniformly square integrable. Due to the uniform integrability, the local martingale M is a true martingale, as shown in [44, Theorem 51, p. 38].

To show the second claim of Proposition 4.1, there are three different ways that lead to the same result.

1. From the definition of the jumps of the martingale $(M_t)_{t\le T}$, we have $\Delta M = M - M_-$, where $M_- := (M_{t-})t \in (0,T]$ is the left-limited process. Thus, ΔM is a predictable process, and its jumps are exhausted by a countable set of predictable stopping times. Since the jumps of the martingale M occur at totally inaccessible stopping times, we have $\Delta M_\tau = 0$ for every $\tau \in \mathscr{T}_0^T$. Note that in the latter equality, we have used the fact that every stopping time can be decomposed into a predictable part and a totally inaccessible part due to the quasi-left continuity of the filtration (see [8, Theorem 41, p. 58], and [8, Theorem 51, p. 62]). Consequently, we obtain $\Delta M = 0 < 1$ and then $S > 0$.
2. Since $(M_t)_{t\le T}$ is a square integrable RCLL martingale, then it is closed by the square integrable random variable M_T. Using [10, Theorem 10, p. 83], we may deduce that for every predicable stopping time τ, we have

$$\Delta M_\tau = -M_{\tau-} = -\mathbb{E}[M_\tau \,|\, \mathscr{F}_{\tau-}] \quad \text{a.s.}$$

   We know by the mentioned properties satisfied by the Azéma martingale M that after a jump τ, we have $M_\tau = 0$. Henceforth, $M_{\tau-} = 0$ for every predictable stopping time τ. By the predictable section theorem [29, Theorem 4.8, p. 115], we infer that $M_- = 0$, which implies $\Delta M = 0$ and then $S > 0$.
3. Using [29, Theorem 3.32, p. 95] , we deuce the existence of an increasing sequence of $\mathbb{F}$-stopping times $\{\tau_n^\varepsilon\}_{n\in\mathbb{N}}$ such that $\{\Delta M \neq 0\} = \{M_- \neq 0\} = \bigcup_{n\in\mathbb{N}} [\![\tau_n^\varepsilon]\!]$, where the sequence τ_n^ε is defined by: $\tau_0^\varepsilon = 0$ and $\tau_{n+1}^\varepsilon = \inf\{s \ge \tau_n^\varepsilon : |M_{s-}| \ge \varepsilon\} \wedge T$ for each given $\varepsilon > 0$. Since M_- has finite left limits on $[0,T]$, $\{\tau_n^\varepsilon\}_{n\in\mathbb{N}}$ has no finite accumulation point. Therefore, $\tau_n^\varepsilon \nearrow T$ $\mathbb{P}$-a.s.. Furthermore, for any $t \in [0,T]$,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le r\le t} (\Delta M_r)^4\right] \\
&\le \sum_{n\in\mathbb{N}} \mathbb{E}\left[\sup_{\tau_n^\varepsilon \le r \le \tau_{n+1}^\varepsilon \wedge t} (\Delta M_r)^4\right] = \sum_{n\in\mathbb{N}} \mathbb{E}\left[\sup_{\tau_n^\varepsilon \le r \le \tau_{n+1}^\varepsilon \wedge t} (M_{r-}\Delta M_r)^2\right] \\
&\le 2\sum_{n\in\mathbb{N}} \mathbb{E}\left[\int_{\tau_n^\varepsilon}^{\tau_{n+1}^\varepsilon \wedge t} M_{u-}^2 d[M,M]_u\right] = \sum_{n\in\mathbb{N}} \mathbb{E}\left[\int_{\tau_n^\varepsilon}^{\tau_{n+1}^\varepsilon \wedge t} M_{u-}^2 d\langle M,M\rangle_u\right]
\end{aligned}$$

$$\leq 2\sum_{n\in\mathbb{N}}\mathbb{E}\left[\int_{\tau_n^\varepsilon}^{\tau_{n+1}^\varepsilon\wedge t} M_{u-}^2\,du\right]\leq 2\varepsilon^2\sum_{n\in\mathbb{N}}\left|\tau_{n+1}^\varepsilon\wedge t-\tau_n^\varepsilon\right|.$$

Due to compactness and the right-continuous, left-limited property of the trajectories of $(M_t)_{t\leq T}$, the ω-section

$$D_\omega:=\{s\geq 0:|\Delta M_s(\omega)|\geq\epsilon\}=\{s\geq 0:|M_{s-}(\omega)|\geq\epsilon\}$$

intersects with $[0,T]$ at most finitely many points. Consequently, there exists a constant C_ϵ such that

$$\sum_{n\in\mathbb{N}}\left|\tau_{n+1}^\varepsilon\wedge t-\tau_n^\varepsilon\right|\leq C_\epsilon(T-t).$$

Therefore, we can obtain

$$\mathbb{E}\left[\sup_{0\leq r\leq t}(\Delta M_r)^4\right]\leq 2C_\epsilon\epsilon^2,\quad\forall\epsilon>0.$$

By taking the limit as $\epsilon\downarrow 0$, we deduce that $\Delta M=0$ on each interval $[0,t]$ for every $t\in[0,T]$. Hence, $\Delta M=0$ and $S>0$ always.

The proof is complete. □

We can now introduce the main result of this paragraph.

Proposition 4.2. *The main characteristics of our market* $(\mathbb{F},\mathbb{P},S^0,S)$ *are*

1. *The market model have no arbitrage opportunities: The risky asset being an* $(\mathbb{F},\mathbb{P})$*-martingale.*
2. *Every contingent claim is redundant: Any bounded* $\mathscr{F}_T$*-measurable random variable* ξ *can be written as a sum of a stochastic integral with respect to* $(S_t)_{t\leq T}$ *and an orthogonal martingale* $(N_t)_{t\leq T}$.

Problem resolution: A RBSDE approach. Prior to delving into the pricing problem, let us introduce the following definition.

Definition 4.1. *A progressively measurable process* $(\phi_t)_{t\leq T}$ *is said to be left-upper semi-continuous along stopping times (LUSCST) if for any stopping time* $\theta\in\mathscr{T}_0^T$ *and any non-decreasing sequence of stopping times* $\{\theta_n\}_{n\in\mathbb{N}}\in(\mathscr{T}_0^T)^{\mathbb{N}}$ *such that* $\theta^n\nearrow\theta$ $\mathbb{P}$*-a.s., we have*

$$\limsup_{n\to\infty}\phi_{\theta_n}\leq\phi_\theta\quad\mathbb{P}\text{-}a.s.$$

It is evident that the above definition incorporates predictable stopping times, which are used to characterize jumps of purely-discontinuous $\mathscr{F}_t$-predictable processes. This fact is described in the following proposition, the proof of which is omitted as it is straightforward.

Proposition 4.3. *Let $(\phi_t)_{t\leq T}$ be an RCLL process. Then $(\varphi_t)_{t\leq T}$ is LUSCST if and only if for each predictable stopping time $\sigma\in\mathscr{T}_0^T$, $\Delta\phi_\sigma\geq 0$ $\mathbb{P}$-a.s.*

We will now consider the pricing of an American contingent claim $\{\mathscr{T}(t,\tau),t\in[0,T]$ and $\tau\in\mathscr{T}_t^T\}$ within the market dynamic model defined by (4.2). Suppose a fixed instant $t\in[0,T]$ and that the buyer chooses to exercise the option at a stopping time $\tau\in\mathscr{T}_t^T$. In this case, the seller must pay the insider the amount

$$\mathfrak{T}(t,\tau)=e^{-R_{t,T}}g(S_T)\mathbb{1}_{\{\tau=T\}}+e^{-R_{t,\tau}}L(S_\tau)\mathbb{1}_{\{\tau<T\}}.$$

Here, $R_{t,T}=\int_t^T r_s ds$ is interpreted as the accumulated interest rate between times t and T.

A reminder that the value process $V:=(V_t)_{t\leq T}$ of an American option is determined by taking the maximum value of the payoff at the time of exercise. Namely,

$$V_t=\underset{\tau\in\mathscr{T}_t^T}{\mathrm{esssup}}\,\mathbb{E}^{\mathscr{F}_t}[\mathfrak{T}(t,\tau)].$$

Through the lens of BSDEs, the pricing of an American option is related to the solution of a linear RBSDE. Specifically, we consider the solution $(Y_t,Z_t,K_t,N_t)_{t\leq T}$ of the following RBSDE:

$$\begin{cases} Y_t=g(S_T)-\displaystyle\int_t^T r_sY_sds+(K_T-K_t)-\int_t^T Z_sdM_s-\int_t^T dN_s, & 0\leq t\leq T,\\ L(S_t)\leq Y_t \quad \mathbb{P}\text{-a.s.}, & 0\leq t\leq T,\\ \displaystyle\int_0^T \big(Y_t-L(S_t)\big)dK_t^c=0, \quad K_t^d=\sum_{0<s\leq t}\big(Y_s-L(S_{s-})\big)^-. \end{cases} \tag{4.4}$$

Let us assume that there exists $p\geq 1$ and $\kappa_p>0$ such that

$$|g(x)|+|L(x)|\leq\kappa_p(1+|x|^p), \quad \forall x\in\mathbb{R}.$$

Under this assumptions, the existence and uniqueness of the solution for the RBSDE (4.4) is guaranteed by Theorem 3.1.

We can now present the main result of this section.

Theorem 4.1. • *The fair price V of the above game contingent claim equals V_0, where $(V_t)_{t\leq T}$ is the right-continuous process such that with $\mathbb{P}$-probability one,*

$$V_t = Y_t = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t^T} \mathbb{E}^{\mathscr{F}_t}[\mathfrak{T}(t,\tau)] \quad \mathbb{P}\text{-a.s.},$$

where the state process $(Y_t)_{t\leq T}$ is the first component of the solution of the RBSDE (4.4).

- *Suppose, in addition, that the process $(L(S_t))_{t\leq T}$ is LUSCST, i.e. in our circumstances $L(S_t)$ may have only positive jumps at predictable points of discontinuity. Then the stopping time $\tau_t^*\in\mathcal{T}_t^T$ defined by $\tau_t^*=\inf\{s\geq t: Y_s = L(S_s)\}\wedge T$ is a saddle point for the above American option which means that*

$$\mathbb{E}^{\mathscr{F}_t}[\mathfrak{T}(t,\tau_t)] \leq \mathbb{E}^{\mathscr{F}_t}[\mathfrak{T}(t,\tau_t^*)] \quad \mathbb{P}\text{-a.s.}, \quad \forall \tau_t\in\mathcal{T}_t^T,$$

and

$$V_t = \mathbb{E}^{\mathscr{F}_t}[\mathfrak{T}(t,\tau_t^*)] \quad \mathbb{P}\text{-a.s.}$$

Proof. Let $(Y_t,Z_t,K_t,N_t)_{t\leq T}$ be a solution of the RBSDE (4.4). Using the integration by parts formula, we obtain for any $\theta\in\mathcal{T}_t^T$,

$$Y_t = e^{-R_{t,\theta}}Y_\theta + \int_t^\theta e^{-R_{t,s}}dK_s - \int_t^\theta e^{-R_{t,s}}Z_s dM_s - \int_t^\theta e^{-R_{t,s}}dN_s. \tag{4.5}$$

Let us fix some $\epsilon>0$ and set

$$\tau_t^\epsilon := \inf\{s\geq t : Y_s \leq L(S_s)+\epsilon\}\wedge T.$$

For $\mathbb{P}$-almost every $\omega\in\Omega$, if $s\in[t,\tau^\epsilon(\omega)[$, then $Y_s > L(S_s)+\epsilon$, thus $Y_s > L(S_s)$. It follows that $\mathbb{P}$-a.s., the continuous process K^c is constant on $[t,\tau_t^\epsilon(\omega)]$ and K^d is constant over $[t,\tau_t^\epsilon(\omega)[$ a.s. Furthermore, since $Y_{\tau_t^\epsilon -}\geq L(S_{\tau_t^\epsilon -})+\epsilon$ a.s. and $\epsilon>0$ it follows that $Y_{\tau_t^\epsilon -} > L(S_{\tau_t^\epsilon -})$, then K^d has no jump at τ_t^ϵ, i.e. $\Delta K^d_{\tau_t^\epsilon}=0$. Hence, $\mathbb{P}$-almost surely, K^d is constant on $[t,\tau_t^\epsilon]$ and then $dK_s=0$ for any $s\in[t,\tau_t^\epsilon]$. On the other hand, using the definitions of τ_t^ϵ and the right-continuity of $L(S_\cdot)$, we obtain

$$Y_{\tau_t^\epsilon} \leq L\big(S_{\tau_t^\epsilon}\big)+\epsilon \quad \text{on } \big\{\tau_t^\epsilon < T\big\}. \tag{4.6}$$

It follows from (4.5) with $\theta=\tau_t^\epsilon$, that

$$Y_t = \mathbb{E}^{\mathscr{F}_t}\big[Y_{\tau_t^\epsilon}\big] \quad \mathbb{P}\text{-a.s.}$$

From (4.6), we also have that $\mathbb{P}$-a.s.

$$\begin{aligned} Y_{\tau_t^\epsilon} &= g(S_T)\mathbb{1}_{\{\tau_t^\epsilon = T\}} + Y_{\tau_t^\epsilon}\mathbb{1}_{\{\tau_t^\epsilon < T\}} \\ &\leq g(S_T)\mathbb{1}_{\{\tau_t^\epsilon = T\}} + \big(L(S_{\tau_t^\epsilon}) + \epsilon\big)\mathbb{1}_{\{\tau_t^\epsilon < T\}}. \end{aligned}$$

Thus,

$$Y_t \leq \mathbb{E}^{\mathscr{F}_t}[\mathfrak{T}(t,\tau_t^\epsilon)] + \epsilon \quad \mathbb{P}\text{-a.s.}$$

As a result, for each $\epsilon > 0$,

$$Y_t \leq \operatorname*{ess\,sup}_{\tau \in \mathscr{T}_t^T} \mathbb{E}^{\mathscr{F}_t}[\mathfrak{T}(t,\tau)] + \epsilon \quad \mathbb{P}\text{-a.s.}$$

Tending ϵ to zero, we get

$$Y_t \leq \operatorname*{ess\,sup}_{\tau \in \mathscr{T}_t^T} \mathbb{E}^{\mathscr{F}_t}[\mathfrak{T}(t,\tau)] = V_t \quad \mathbb{P}\text{-a.s.} \tag{4.7}$$

On the other hand, going back to the BSDE (4.5), we deduce that, for each $\theta \in \mathscr{T}_t^T$,

$$Y_t \geq \mathbb{E}^{\mathscr{F}_t}\left[e^{-R_{t,\theta}} Y_\theta\right],$$

since $\int_t^\theta e^{-R_{t,s}} dK_s \geq 0$. Then, using the following inequality:

$$e^{-R_{t,\theta}} Y_\theta \geq e^{-R_{t,T}} g(S_T)\mathbb{1}_{\{\theta = T\}} + e^{-R_{t,\theta}} L(S_\theta)\mathbb{1}_{\{\theta < T\}} \quad \mathbb{P}\text{-a.s.},$$

we deduce that, for every $\theta \in \mathscr{T}_t^T$,

$$Y_t \geq \mathbb{E}^{\mathscr{F}_t}[\mathfrak{T}(\theta,t)].$$

Thus, from the very definition of the essential supremum, we can easily conclude that

$$Y_t \geq \operatorname*{ess\,sup}_{\tau \in \mathscr{T}_t^T} \mathbb{E}^{\mathscr{F}_t}[\mathfrak{T}(t,\tau)] = V_t \quad \mathbb{P}\text{-a.s.} \tag{4.8}$$

In force of inequalities (4.7) and (4.8), we obtain that $Y_t = V_t$ $\mathbb{P}$-a.s.

Suppose now that $L(S_t)$ is LUSCST. Then using Proposition 4.3 implies that the reflection process $(K_t)_{t \leq T}$ is continuous on $[0,T]$. Indeed, let $\sigma \in \mathscr{T}_0^T$ be a negative predictable jump time of the lower barrier $(L(S_t))_{t \leq T}$. From Proposition 4.3, we get

$$\Delta K_\sigma^d = \big(L(S_{\sigma-}) - Y_\sigma\big)^+ \mathbb{1}_{\{Y_{\sigma-} = L(S_{\sigma-})\}} \leq \big(L(S_\sigma) - Y_\sigma\big)^+ \mathbb{1}_{\{Y_{\sigma-} = L(S_{\sigma-})\}} = 0.$$

Hence, $\Delta K^d_\sigma = 0$ $\mathbb{P}$-a.s. and this holds for any predictable stopping time σ. Consequently, using section theorem (see [29, Theorem 4.10, p. 116]), we deduce that K is a continuous process.

Again, thanks to the right continuity of Y and $L(S_\cdot)$, we get

$$Y_{\tau_t^*} = L\big(S_{\tau_t^*}\big) \quad \text{on } \{\tau_t^* < T\}.$$

By definition of τ_t^*, we have $Y_s > L(S_s)$ $\forall s \in [t, \tau_t^*[$ $\mathbb{P}$-a.s. Since $(Y_t, Z_t, K_t^+, K_t^-)_{t\leq T}$ is a solution of the DRBSDE (4.4), thus K is constant on $[t, \tau_t^*]$. Hence, we obtain $Y_t = \mathbb{E}^{\mathscr{F}_t}[\mathfrak{T}_t(\tau_t^*, \sigma_t^*)]$ a.s. Using the same arguments as in (4.8), one can demonstrate that for each $\tau_t \in \mathscr{T}_t^T$,

$$Y_t = \mathbb{E}^{\mathscr{F}_t}[\mathfrak{T}(t, \tau_t^*)] \geq \mathbb{E}^{\mathscr{F}_t}[\mathfrak{T}(t, \tau_t)] \quad \mathbb{P}\text{-a.s.},$$

which yields that τ_t^* is a saddle point for the American contingent claim. □

Perspective and future work

In light of the literature on irregular obstacles (see, e.g. [3–6, 18, 23, 24, 33, 35]), it is natural to extend our analysis to reflected BSDEs driven by general RCLL martingales with a completely irregular barrier. A promising direction is to establish existence and uniqueness under stochastic Lipschitz conditions on the driver in this broader setting, and to analyze the associated optimal stopping problem via an appropriate nonlinear expectation induced by classical BSDE driven by the general RCLL martingale. Such an extension would connect the present results to the irregular-obstacle framework and further develop the existing literature on this topic.

Appendix A. Special BSDE without reflection driven by an RCLL martingale

We provide a specific case of existence and uniqueness for BSDEs driven by the RCLL martingale M in this section when the coefficient depends only on y. Consider the following BSDE:

$$Y_t = \xi + \int_t^T f(s, Y_s)ds + \int_t^T g(s)d\langle M\rangle_s - \int_t^T Z_s dM_s - \int_t^T dN_s, \quad 0 \leq t \leq T. \tag{A.1}$$

Theorem A.1 (Existence and Uniqueness Result). *Assume that*

(i) $\xi \in \mathbb{L}^2_\beta$.

(ii) $\dfrac{g(\cdot)}{\alpha_\cdot} \in \mathscr{H}^2_\beta$ *and* $f(\cdot,0) \in \mathscr{C}^2_\beta$.

(iii) The driver f *is uniformly Lipschitz continuous with respect to* y*, i.e. there exists a positive constant* κ *such that, almost every* (ω,t)*, for all* $y, y' \in \mathbb{R}$,

$$|f(t,y) - f(t,y')| \leq \kappa |y - y'|.$$

Then, the BSDE (A.1) *admit a unique solution*

$$(Y,Z,N) \in \left(\mathscr{S}^2_\beta \cap \mathscr{S}^{2,\alpha}_\beta \cap \mathscr{C}^2_\beta \right) \times \mathscr{H}^2_\beta \times \mathscr{M}^2_\beta.$$

Theorem A.2 (Comparison Theorem). *Let* $(Y^1,Z^1,N^1),(Y^2,Z^2,N^2)$ *be solutions of BSDE* (A.1) *associated with parameters* (ξ^1,f^1,g) *and* (ξ^2,f^2,g)*, respectively. Assume that* $\xi^1 \leq \xi^2$ *and for any* $t \geq 0$, $f^1(t,y) \leq f^2(t,y)$*, for all* $y \in \mathbb{R}$ $\mathbb{P}$*-a.s. Then* $Y^1 \leq Y^2$ *a.s.*

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