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Hypersonic Limit for Steady Compressible Euler Flows Passing Straight Cones

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Abstract. We investigate the hypersonic limit for steady, uniform, and compressible polytropic gas passing a symmetric straight cone. By considering Radon measure solutions, we show that as the Mach number of the upstream flow tends to infinity, the measures associated with the weak entropy solution containing an attached shock ahead of the cone converge vaguely to the measures associated with a Radon measure solution to the conical hypersonic-limit flow. This justifies the Newtonian sine-squared pressure law for cones in hypersonic aerodynamics. For Chaplygin gas, assuming that the Mach number of the incoming flow is less than a finite critical value, we demonstrate that the vertex angle of the leading shock is independent of the conical body's vertex angle and is totally determined by the incoming flow's Mach number. If the Mach number exceeds the critical value, we explicitly construct a Radon measure solution with a concentration boundary layer.

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Key words: Compressible Euler equations, shock wave, conical flow, hypersonic limit, Radon measure solution.

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1 Introduction

This paper pertains to uniform supersonic flows passing a straight right-circular cone $\mathfrak{C}$, where the cone's half vertex-angle θ_0 is less than the critical detach angle and sonic angle. In this scenario, a conical shock attaches to the cone's vertex, and the flow between the shock-front and the cone is supersonic. In addition to demonstrating the existence of such weak solutions by solving an inverse problem, we are interested in the hypersonic limit and justify the celebrated Newtonian sine-squared pressure law for cones. To describe the hypersonic-limit flow field, which involves an infinitely-thin boundary layer exhibiting concentration of mass and momentum, an appropriate concept of Radon measure solutions to the compressible Euler equations is required.

We assume that the flow field is governed by the steady compressible Euler system

$$\begin{cases} \mathrm{Div}(\rho V)=0, \\ \mathrm{Div}(\rho V\otimes V)+\mathrm{Grad}\, p=0, \\ \mathrm{Div}(\rho E V)=0. \end{cases} \tag{1.1}$$

Here Div and Grad are the divergence and gradient operator of the Euclidean space $\mathbb{R}^3$ respectively. The unknowns ρ,E and $V=(u_1,u_2,u_3)^\top$ represent respectively the density of mass, the total enthalpy per unit mass, and the flow's velocity in $\mathbb{R}^3$. For a polytropic gas, the unknown scalar pressure p is given by the state equation

$$p=\frac{\epsilon}{\epsilon+1}\rho\cdot\left(E-\frac{1}{2}|V|^2\right) \tag{1.2}$$

with $\gamma\doteq\epsilon+1>1$ being the adiabatic exponent. Then the sound speed is $c\doteq\sqrt{\gamma p/\rho}$, and the Mach number is $M\doteq|V|/c$. It will be shown later that after suitable scalings, the hypersonic limit is the limit $\epsilon\to 0+$. The case of Chaplygin gas will be considered in Section 5.

The flow is supposed to be given and moves uniformly at supersonic speed from the left half-space

$$(\rho,V,E)=\big(\rho_\infty,(V_\infty,0,0)^\top,E_\infty\big) \quad \text{in } \big\{x=(x_1,x_2,x_3)\in\mathbb{R}^3 : x_1\le 0\big\}, \tag{1.3}$$

and is subjected to the slip boundary condition

$$V\cdot\vec{n}|_{\mathcal{C}}=0 \tag{1.4}$$

on the boundary

$$\mathcal{C}\doteq\partial\mathfrak{C}=\big\{x\in\mathbb{R}^3 : x_1>0,\, x_2^2+x_3^2=x_1^2\tan\theta_0^2\big\} \tag{1.5}$$