Commun. Math. Anal. Appl.
doi: 10.4208/cmaa.2024-0009

Vol. **3**, No. 2, pp. 168-198
May 2024

The Incompressible Limit of the Equations of Compressible Ideal Magneto-Hydrodynamics with Perfectly Conducting Boundary

Paolo Secchi*

INdAM Unit & Department of Civil, Environmental, Architectural Engineering and Mathematics (DICATAM), University of Brescia, Via Valotti 9, 25133 Brescia, Italy

Received 5 April 2024; Accepted 4 May 2024

Abstract. We consider the initial-boundary value problem in the halfspace for the system of equations of ideal magneto-hydrodynamics (MHD) with a perfectly conducting wall boundary condition. We show the convergence of solutions to the solution of the equations of incompressible MHD as the Mach number goes to zero. Because of the characteristic boundary, where a loss of regularity in the normal direction to the boundary may occur, the convergence is shown in suitable anisotropic Sobolev spaces which take account of the singular behavior at the boundary.

AMS subject classifications: 35L50, 35Q35, 76M45, 76W05

Key words: Compressible ideal magneto-hydrodynamics, Mach number, incompressible limit, singular limit, perfectly conducting wall.

1 Introduction

We consider the equations of ideal magneto-hydrodynamics for the motion of an electrically conducting fluid, where "ideal" means that the effect of viscosity and electrical resistivity is neglected (see [7])

*Corresponding author. *Email address:* paolo.secchi@unibs.it (P. Secchi)

$$\rho_p(p^\lambda)\big(\partial_t+v^\lambda\cdot\nabla\big)p^\lambda+\rho^\lambda\nabla\cdot v^\lambda=0, \tag{1.1a}$$

$$\rho^\lambda\big(\partial_t+(v^\lambda\cdot\nabla)\big)v^\lambda+\lambda^2\nabla p^\lambda+\mu H^\lambda\times(\nabla\times H^\lambda)=0, \tag{1.1b}$$

$$\big(\partial_t+(v^\lambda\cdot\nabla)\big)H^\lambda-(H^\lambda\cdot\nabla)v^\lambda+H^\lambda\nabla\cdot v^\lambda=0. \tag{1.1c}$$

Here the pressure $p^\lambda=p^\lambda(t,x)$, the velocity field $v^\lambda=v^\lambda(t,x)=(v_1^\lambda,v_2^\lambda,v_3^\lambda)$, the magnetic field $H^\lambda=H^\lambda(t,x)=(H_1^\lambda,H_2^\lambda,H_3^\lambda)$ are unknown functions of time t and space variables $x=(x_1,x_2,x_3)$. The density ρ^λ is given by the equation of state $\rho^\lambda=\rho(p^\lambda)$ where $\rho>0$ and $\partial\rho/\partial p\equiv\rho_p>0$ for $p>0$. The magnetic permeability μ is set equal to 1 without loss of generality. The coefficient λ is essentially the inverse of the Mach number. We denote $\partial_t=\partial/\partial t,\partial_i=\partial/\partial x_i,\nabla=(\partial_1,\partial_2,\partial_3)$ and use the conventional notations of vector analysis. The system (1.1) is supplemented with the divergence constraint

$$\nabla\cdot H^\lambda=0 \tag{1.2}$$

on the initial data.

We study the initial-boundary value problem corresponding to a perfectly conducting wall boundary condition. Set $\Omega=\mathbb{R}^3_+=\{x_1>0\}$ and let us denote its boundary by Γ. We also denote $Q_T=(0,T)\times\Omega,\Sigma_T=(0,T)\times\Gamma$ and denote by $\nu=(-1,0,0)$ the unit outward normal to Γ. We are interested in the study of the initial-boundary value problem under the boundary conditions

$$v^\lambda\cdot\nu=0,\quad H^\lambda\cdot\nu=0\quad\text{on }\Sigma_T. \tag{1.3}$$

System (1.1)-(1.3) is supplemented with initial conditions

$$(p^\lambda,v^\lambda,H^\lambda)_{|t=0}=\big(p_0^\lambda,v_0^\lambda,H_0^\lambda\big)\quad\text{in }\Omega. \tag{1.4}$$

We study the singular limit as $\lambda\to+\infty$. The limit equations to system (1.1)-(1.3) are

$$\overline{\rho}\big(\partial_t+(w\cdot\nabla)\big)w+\nabla\pi+B\times(\nabla\times B)=0, \tag{1.5a}$$

$$\partial_t B+(w\cdot\nabla)B-(B\cdot\nabla)w=0, \tag{1.5b}$$

$$\nabla\cdot w=0,\quad \nabla\cdot H=0\quad\text{in }Q_T, \tag{1.5c}$$

$$w\cdot\nu=0,\quad B\cdot\nu=0\quad\text{on }\Sigma_T, \tag{1.5d}$$

$$(w,B)_{|t=0}=(w_0,B_0)\quad\text{in }\Omega, \tag{1.5e}$$

where $\overline{\rho}=\rho(0)$ and w_0 is such that $\nabla\cdot w_0=0$ in Ω and $w_0\cdot\nu=0$ on Γ and analogously for B_0.