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Relativistic Fluid Flows in a Bounded Domain

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In honor of Professor Gui-Qiang Chen 60th birthday.

Abstract. We analyze a boundary value problem for the flow of a relativistic fluid confined in a bounded domain. An important point is that we base our problem on the Lagrange coordinates of the fluid which we define here, for the first time in the $(1+1)$-dimensional case. The Lagrange coordinates for the $1+3$ relativistic flow was established in [Dias and Frid, Comm. Math. Anal. Appl. 2(3) (2023)].

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1 Introduction

We consider the boundary value problem corresponding to a $(1+1)$-dimensional relativistic fluid confined to a bounded domain. Namely, we consider the initial-boundary value problem for the one-dimensional relativistic Euler equations

$$\begin{cases} \dfrac{\partial}{\partial t}\left(\dfrac{p+\rho c^2}{c^2}\dfrac{u^2}{c^2-u^2}+\rho\right)+\dfrac{\partial}{\partial x}\left((p+\rho c^2)\dfrac{u}{c^2-u^2}\right)=0, \\ \dfrac{\partial}{\partial t}\left((p+\rho c^2)\dfrac{u}{c^2-u^2}\right)+\dfrac{\partial}{\partial x}\left((p+\rho c^2)\dfrac{u^2}{c^2-u^2}+p\right)=0, \\ (t,x)\in(0,\infty)\times(0,1). \end{cases} \tag{1.1}$$

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Here, as usual, ρ is the density, $p = p(\rho)$ is the pressure, u is the velocity, c is the light speed. The physical domain is $\rho > 0, 0 < p'(\rho) < c^2$ and $|u| < c$. We impose to (1.1) the following initial and boundary conditions:

$$\begin{cases} \rho_{re}(0,x) = \rho_{re,0}(x), \\ u_{re}(0,x) = u_{re,0}(x), \end{cases} \tag{1.2}$$

and

$$u_{re}(t,0) = u_{re}(t,1) = 0, \tag{1.3}$$

where we define

$$\rho_{re} = \frac{pu^2 + c^4\rho}{c^2(c^2 - u^2)}, \tag{1.4}$$

$$u_{re} = \frac{c^2 u(p + \rho c^2)}{pu^2 + c^4\rho}. \tag{1.5}$$

We remark that $u_{re} = 0$ if and only if $u = 0$.

We begin by establishing the fact that the transformation $(\rho, u) \mapsto (\rho_{re}, u_{re})$ is one-to-one in the physical domain $\rho > 0, 0 < p'(\rho) < c^2, |u| < c$. It is basically an extension of [9, Proposition 1] (see also [3]). We defer its proof to Section 5.

Proposition 1.1. *The mapping $(\rho, u) \mapsto (\rho_{re}, u_{re})$ is a one-to-one local diffeomorphism on the physical domain $\rho > 0, 0 < p'(\rho) < c^2, |u| < c$.*

The contents of this paper are as follows. After this brief introduction, Section 1, in Section 2, we introduce and analyze the $(1+1)$-dimensional Lagrange transformation. In Section 3, we recast the initial-boundary value problem (1.1)-(1.3) in Lagrange coordinates and also some its main properties, deferring the proofs to the last section. In Section 4, we give the proof of the existence of a globally defined admissible solution of the problem (1.1)-(1.3). Finally, in Section 5, we give the proofs of Propositions 1.1, 3.1 and 3.2.

2 The Lagrange coordinates

We next define the Lagrange coordinates of the $1+1$ relativistic flow governed by (1.1). We recall that the Lagrange coordinates for the $1+3$ relativistic flow was established in [5]. Observe that the system (1.1) may be written in the form

$$\begin{cases} (\rho_{re})_t + (\rho_{re} u_{re})_x = 0, \\ (\rho_{re} u_{re})_t + (\rho_{re} u_{re} u + p)_x = 0. \end{cases} \tag{2.1}$$