Commun. Math. Anal. Appl.
doi: 10.4208/cmaa.2024-0023

Vol. **3**, No. 4, pp. 532-557
November 2024

Some Recent Developments on Isometric Immersions via Compensated Compactness and Gauge Transforms

Siran Li*

School of Mathematical Sciences & CMA-Shanghai, Shanghai Jiao Tong University, Shanghai 200240, China.

Received 14 September 2024; Accepted 22 October 2024

Dedicated to Professor Gui-Qiang G. Chen on the occasion of his 60^{th} birthday, with gratitude and admiration.

Abstract. We survey recent developments on the analysis of Gauss-Codazzi-Ricci equations, the first-order PDE system arising from the classical problem of isometric immersions in differential geometry, especially in the regime of low Sobolev regularity. Such equations are not purely elliptic, parabolic, or hyperbolic in general, hence calling for analytical tools for PDEs of mixed types. We discuss various recent contributions – in line with the pioneering works [Chen *et al.*, Proc. Amer. Math. Soc. 138 (2010), Commun. Math. Phys. 294 (2010)] – on the weak continuity of Gauss-Codazzi-Ricci equations, the weak stability of isometric immersions, and the fundamental theorem of submanifold theory with low regularity. Two mixed-type PDE techniques are emphasised throughout these developments: the method of compensated compactness and the theory of Coulomb-Uhlenbeck gauges.

AMS subject classifications: 35M10, 35M30, 74B20, 53Z05, 35R01, 53C24, 53A07, 53C42, 53Z05

Key words: Isometric immersions, nonlinear elasticity, Gauss-Codazzi-Ricci equations, compensated compactness, gauge theory.

*Corresponding author. *Email address:* `siran.li@sjtu.edu.cn` (S. Li)

1 Introduction

Isometric immersions and/or embeddings of Riemannian manifolds have long been a central topic in global analysis and geometric PDEs. Studies on the existence, uniqueness, stability, and regularity of isometric immersions (embeddings) of Riemannian manifolds into Euclidean spaces or general ambient manifolds abound in the literature. These works have partially shaped the landscapes of analysis, geometry, and PDE nowadays, and have found wide-ranged, in-depth applications in mathematical relatively, continuum mechanics, and mathematical biology, etc. Let us mention here the groundbreaking works of historical significance by Cartan [13], Weyl [83], Aleksandroff [2], Nirenberg [64], Nash [62, 63], and Pogorelov [65], this list is by no means exhaustive. We also refer the reader to Han and Hong [38] for a comprehensive survey, and the problem section in Yau [85] for related questions.

This contribution focuses on isometric immersions of weak regularities. Consider a compact Riemannian manifold $(\mathcal{M},g)$ of dimension $n\geq 2$. A mapping $\Phi:\mathcal{M}\to\mathbb{R}^{n+k}$ is an immersion if $d\Phi:T\mathcal{M}\to T\mathbb{R}^{n+k}$ is injective and an isometry if $\Phi^{\#}\delta=g$, where δ denotes the Euclidean metric on $\mathbb{R}^{n+k}$ and $\Phi^{\#}$ the pullback under Φ. By "weak regularity" we usually mean $\Phi\in W^{2,p}$ and $g\in W^{1,p}\cap L^{\infty}$ with some $p\in[1,\infty]$. We emphasise the PDE approach throughout: all the results discussed in this survey are obtained by analysing weak (distributional) solutions to the Gauss-Codazzi-Ricci equations and their variants associated to the isometric immersions.

1.1 The PDE system

Gauss-Codazzi-Ricci equations are the compatibility equations of curvatures for the existence of an isometric immersion $\Phi:(\mathcal{M}^n,g)\to\mathbb{R}^{n+k}$. In geometrical terms, the intrinsic geometry of Φ is given by the metric g, while the extrinsic geometry is given by the second fundamental form II and the normal connection $\nabla^{\perp}$ (i.e. the orthogonal projection of the Levi-Civita connection on $\mathbb{R}^{n+k}$ to the normal bundle of Φ, which is trivial when the codimension $k=1$). The curvature components in the tangential and normal directions of Φ together constitute the flat Riemann curvature on $\mathbb{R}^{n+k}$. This gives rise to the Gauss-Codazzi-Ricci equations associated to $\Phi:(\mathcal{M}^n,g)\to\mathbb{R}^{n+k}$ [31]

$$\delta\big(\mathrm{II}(X,Z),\mathrm{II}(Y,W)\big)-\delta\big(\mathrm{II}(X,W),\mathrm{II}(Y,Z)\big)=R(X,Y,Z,W), \tag{1.1}$$

$$\overline{\nabla}_Y\mathrm{II}(X,Z)-\overline{\nabla}_X\mathrm{II}(Y,Z)=0, \tag{1.2}$$

$$g\big([\mathcal{S}_\eta,\mathcal{S}_\zeta]X,Y\big)=R^E(X,Y,\eta,\zeta) \tag{1.3}$$