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Global Well-Posedness and Optimal Time Decay Rates of Solutions to the Pressureless Euler-Navier-Stokes System

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Dedicated to Professor Gui-Qiang Chen on the occasion of his 60th birthday.

Abstract. In this paper, we present a new framework for the global well-posedness and large-time behavior of a two-phase flow system, which consists of the pressureless Euler equations and incompressible Navier-Stokes equations coupled through the drag force. To overcome the difficulties arising from the absence of the pressure term in the Euler equations, we establish the time decay estimates of the high-order derivative of the velocity to obtain uniform estimates of the fluid density. The upper bound decay rates are obtained by designing a new functional and the lower bound decay rates are achieved by selecting specific initial data. Moreover, the upper bound decay rates are the same order as the lower one. Therefore, the time decay rates are optimal. When the fluid density in the pressureless Euler flow vanishes, the system is reduced into an incompressible Navier-Stokes flow. In this case, our works coincide with the classical results by Schonbek [J. Amer. Math. Soc. 4 (1991)], which can be regarded as a generalization from a single fluid model to the two-phase fluid one.

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1 Introduction

This paper is concerned with the global well-posedness and large-time behavior of a coupled hydrodynamic system consisting of the pressureless Euler equations and incompressible Navier-Stokes equations coupled through the drag force, which reads as

$$\begin{cases} \rho_t+\mathrm{div}(\rho u)=0, & \text{(1.1a)} \\ (\rho u)_t+\mathrm{div}(\rho u\otimes u)=-\rho(u-v), & \text{(1.1b)} \\ v_t+v\cdot\nabla v+\nabla P=\Delta v+\rho(u-v), & \text{(1.1c)} \\ \mathrm{div} v=0 & \text{(1.1d)} \end{cases}$$

with the initial data

$$(\rho,u,v)|_{t=0}=(\rho_0,u_0,v_0). \tag{1.2}$$

The unknown functions $\rho=\rho(t,x)$ and $u=u(t,x)$ denote the density and velocity of the pressureless Euler fluid flow, $v=v(t,x)$ and $P=P(t,x)$ represent the velocity and pressure of the incompressible Navier-Stokes fluid flow.

This coupled pressureless Euler-Navier-Stokes (E-NS) system can be formally derived from the Vlasov-Navier-Stokes system, which describes the behavior of a large cloud of particles interacting with the incompressible fluid, in the case of mono-kinetic particle distributions. The details of the derivation can be referred to [7]. The main system (1.1) is closely related to the kinetic-fluid models, which have received increasing attention due to its wide range of applications, for instance, including medicine, biotechnology, combustion in diesel engines, and atmospheric pollution [1,2,18,21,23].

There has been important progress made recently on the well-posedness and dynamic behaviors of the solutions to the Euler-Navier-Stokes system and related models, refer to [4,5,12,14,17] and the references therein. Concerning the Euler system coupled with compressible Navier-Stokes equations, Choi [6] proved the global well-posedness and exponential time decay rates of the classical solution in the periodic domain. Later, Wu *et al.* [24] obtained the large-time behavior in $\mathbb{R}^3$ and Wu and Zhou [25] studied the pointwise space-time behavior of the Cauchy problem to this system.