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Logarithmic Upper Bound for Solutions of Degenerate Parabolic Equation

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Abstract. In this note, we consider the following degenerate parabolic equation studied in [F. Chiarenza and R. Serapioni, *Degenerate parabolic equations and Harnack inequality*, Ann. Mat. Pura Appl. 137 (1984)] i.e.,

$$\begin{cases} \dfrac{\partial u}{\partial t}-\dfrac{\partial}{\partial x_i}\left(a_{ij}(x,t)\dfrac{\partial u}{\partial x_j}\right)=-\operatorname{div} \boldsymbol{f} & \text{in } \Omega\times(0,T), \\ u=0 & \text{on } \partial\Omega\times(0,T), \\ u(x,0)=u_0(x) & \text{in } \Omega, \end{cases}$$

where $\boldsymbol{f}=(f^1,\cdots,f^n)$ and Ω is a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary, $n\geq 2$ and $T>0$. In this paper, we apply Moser iteration argument to build up the explicit relationship among the coefficients $a_{i,j}(x,t)$, $\boldsymbol{f}$ and the maximum norm of the solution. Meanwhile, we also find that the weighed Lebesgue space $L^{2l/(l-1)}$ to which $\boldsymbol{f}$ belongs is essentially sharp in order to establish local boundedness of the solution. Here the definition of l is found in Lemma 2.3. Our results cover the well-known results.

AMS subject classifications: 35K20, 35D30, 35B50

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1 Introduction

The purpose of this paper is to analyze the relation between the maximum norm of the solution and vector function $\boldsymbol{f}$. Namely, we discuss the following Cauchy-

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Dirichlet problem:

$$\begin{cases} \mathcal{L}u := \dfrac{\partial u}{\partial t} - \dfrac{\partial}{\partial x_i}\left(a_{ij}(x,t)\dfrac{\partial u}{\partial x_j}\right) = -\left(f^i\right)_{x_i} & \text{in } \Omega\times(0,T), \\ u=0 & \text{on } \partial\Omega\times(0,T), \\ u(x,0)=u_0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary, $n\geq 2$ and $T>0$. The coefficients $a_{ij}(x,t)$, the nonlinear term f^i and the initial date u_0 satisfy the following assumptions:

(H1) The coefficients $a_{ij}(x,t)$ are non-negative measurable functions, satisfying

$$a_{ij}(x,t)=a_{ji}(x,t), \quad i,j=1,\ldots,n, \quad x\in\Omega, \quad t>0.$$

(H2) There exists $\lambda>0$ such that

$$\frac{1}{\lambda}\omega(x,t)|\boldsymbol{\xi}|^2 \leq a_{ij}(x,t)\xi_i\xi_j \leq \lambda\omega(x,t)|\boldsymbol{\xi}|^2$$

for all $\boldsymbol{\xi}\in\mathbb{R}^n$ and a.e. $(x,t)\in Q_T:=\Omega\times[0,T]$, where $\omega(x,t)$ is an A_2 weight in $\mathbb{R}^n$ uniformly with respect to t in $(0,T)$, and an A_2 weight in $(0,T)$ uniformly with respect to x in Ω. See more details in Section 2.

(H3) The nonlinear term $\boldsymbol{f}=(f^1,\cdots,f^n)$ satisfies

$$\frac{f^i}{\omega}\in L^r(Q_T;\omega) \quad \text{with} \quad r>\frac{2l}{l-1},$$

where l is the same as in Lemma 2.3.

(H4) The initial date u_0 satisfies

$$\sup_{\Omega}|u_0(x)|=K_0<+\infty.$$

It is well-known, when $\omega^{-1}(x,t)$ and $\omega(x,t)$ are essentially bounded, the problem (1.1) reduces the uniform parabolic problem. In this situation, the study dates back to works of Moser [19, 20]. For more pioneer works, we may refer to the monographs by Ladyženskaja *et al.* [14] and Lieberman [16].

On the contrary, if $\omega^{-1}(x,t)$ is unbounded, the problem (1.1) is degenerate. In particular, when the weight is time independent i.e., $\omega(x,t)=\omega(x)$, Chiarenza