Commun. Math. Res.
doi: 10.4208/cmr.2025-0023

Vol. **41**, No. 3, pp. 225-249
September 2025

The Spectral Einstein Functional for the Nonminimal de Rham-Hodge Operator

Hongfeng Li[1] and Yong Wang[2,*]

[1] *School of Mathematics and Systems Science, Shenyang Normal University, Shenyang 110034, P.R. China.*
[2] *School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, P.R. China.*

Received 30 May 2025; Accepted 9 August 2025

Abstract. In this paper, we give the definitions of the non-self-adjoint spectral triple and its spectral Einstein functional. We compute the spectral Einstein functional associated with the nonminimal de Rham-Hodge operator on even-dimensional compact manifolds without boundary. Finally, several examples of the non-self-adjoint spectral triple are listed.

AMS subject classifications: 53C40, 53C42

Key words: The nonminimal de Rham-Hodge operator, non-self-adjoint spectral Einstein functional, noncommutative residue.

1 Introduction

Until now, many geometers have studied the noncommutative residue, which is of great importance to the study of noncommutative geometry. Connes [2] showed that the noncommutative residue on a compact manifold M coincided with the Dixmier's trace on pseudo-differential operators of order $-\mathrm{dim}M$. Hence, the noncommutative residue can be used as an integral in noncommutative geometry and become an important tool of noncommutative geometry. In [4],

*Corresponding author. *Email addresses:* lihf728@nenu.edu.cn (H. Li), wangy581@nenu.edu.cn (Y. Wang)

Connes used the noncommutative residue to derive a conformal 4-dimensional Polyakov action analogy. A few years ago, Connes made a challenging observation that the noncommutative residue of the square of the inverse of the Dirac operator was proportional to the Einstein-Hilbert action, which is called the Kastler-Kalau-Walze type theorem. Kastler [12] gave a brute-force proof of this theorem. Kalau and Walze [11] proved this theorem in the normal coordinates system simultaneously. Using the theory of noncommutative residue proposed by Wodzicki, Fedosov *et al.* [8] constructed the noncommutative residue of the classical elemental algebra of the Boutet de Monvel calculus on compact manifolds of dimension $n>2$.

Using elliptic pseudo-differential operators and the noncommutative residue is a natural way to study the spectral Einstein functional and the operator-theoretic interpretation of the gravitational action on bounded manifolds. Concerning the Dirac operator and the signature operator, Wang carried out the computation of noncommutative residues and succeeded in proving the Kastler-Kalau-Walze type theorem for manifolds with boundary [17–19]. Figueroa *et al.* [9] introduced a noncommutative integral based on the noncommutative residue. Pfäffle and Stephan [13] considered orthogonal connections with arbitrary torsion on compact Riemannian manifolds and computed the spectral action. Dabrowski *et al.* [6] defined the spectral Einstein functional for a general spectral triple and for the noncommutative torus, they computed the spectral Einstein functional. Wang *et al.* [16] gave some new spectral functionals which are the extension of spectral functionals to the noncommutative realm with torsion, and related them to the noncommutative residue for manifolds with boundary about Dirac operators with torsion. Dabrowski *et al.* [7] examined the metric and Einstein bilinear functionals of differential forms for the Hodge-Dirac operator $d+d^*$ on an oriented, closed, even-dimensional Riemannian manifold. Hong and Wang [10] computed the spectral Einstein functional associated with the Dirac operator with torsion on even-dimensional spin manifolds without boundary. Wu and Wang [21] computed the spectral Einstein functional for the Witten deformation $d+d^*+\widehat{c}(V)$ on even-dimensional Riemannian manifolds without boundary. Wu and Wang [20] obtained the Lichnerowicz type formula for the Witten deformation of the nonminimal de Rham-Hodge operator, and the gravitational action in the case of n-dimensional compact manifolds without boundary. Finally, the authors gave the proof of Kastler-Kalau-Walze type theorems for the Witten deformation of the nonminimal de Rham-Hodge operator on 4,6-dimensional oriented compact manifolds with boundary. In [15], the authors proved Kastler-Kalau-Walze type theorems associated with nonminimal de Rham-Hodge operators on compact manifolds with boundary.