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Dynamical Stability of Transonic Shock Solutions to Non-Isentropic Euler Equations

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Abstract. In this paper, we investigate the dynamical stability of transonic shock solutions for the full compressible Euler system in a two dimensional nozzle with a symmetric divergent part. Building upon the existence and uniqueness results for steady symmetric transonic shock solutions to the non-isentropic Euler system established in [Z.P. Xin and H.C. Yin, *The transonic shock in a nozzle, 2-D and 3-D complete Euler systems*, J. Differential Equations 245 (2008)], we prove the dynamical stability of the transonic shock solutions under small perturbations. More precisely, if the initial unsteady transonic flow is located in the symmetric divergent part of the nozzle and the flow is a symmetric small perturbation of the steady transonic flow, we use the characteristic method to establish the dynamical stability.

AMS subject classifications: 35B35, 35L65, 76H05
Key words: Euler equation, transonic shock, dynamical stability.

1 Introduction

In this paper, we investigate the two dimensional full Euler system

$$\begin{cases} \rho_t + \mathrm{div}(\rho\mathbf{u}) = 0, \\ (\rho\mathbf{u})_t + \mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u} + p\mathbf{I}) = 0, \\ (\rho\mathcal{E})_t + \mathrm{div}(\rho\mathcal{E}\mathbf{u} + p\mathbf{u}) = 0, \end{cases}$$

where $\mathbf{u} = (u,v)$, ρ, p and $\mathcal{E}$ represent the velocity, the density, the pressure and

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the energy, respectively. We consider ideal polytropic gas, the equations of state and the total energy are given by

$$p = e^S \rho^\gamma, \quad \mathcal{E} = \frac{|u|^2}{2} + \frac{p}{(\gamma-1)\rho},$$

where $\gamma > 1$ is the adiabatic exponent and S is the entropy.

Courant and Friedrichs [8] proposed a conjecture about transonic shocks in de Laval nozzles: given an appropriate pressure at the nozzle exit, a subsonic inflow would first accelerate to sonic at the throat, then transition to supersonic flow in the diverging part, and finally become subsonic flow across a shock front. There have been a few theoretical analysis of the transonic shock flows. For the steady Euler system in straight nozzles, Chen [4] established the existence and stability of a transonic shock solutions under the assumption that the shock goes through a fixed point. Chen and Feldman [2] proved the existence of potential transonic shocks with the additional constraint that the shock front passes through a given point. For the steady Euler system in de Laval nozzles, Xin and Yin [17,18] proved the unique existence and stability of transonic shocks in two dimensional and three dimensional nozzles under the restriction that the shock goes through the fixed point. Chen and Feldman [3] investigated the existence and stability of steady potential compressible flows with transonic shocks in infinite nozzles of arbitrary cross-sections. Subsequently, Bae and Feldman [1] studied transonic shock problems for non-isentropic potential flow in multidimensional divergent nozzles with arbitrary smooth cross-sections. Based on the theory in [3], Wang and Xie [14] constructed a single transonic shock wave pattern in an infinite nozzle which asymptotically converges to a cylinder. They also obtained the stability of the uniform transonic wave, while also proving the stability of the uniform transonic shock. Additional significant results on transonic shocks for the Euler system have been achieved in [5–7, 9, 13, 15, 16] and the references therein.

For quasi-one-dimensional Euler system, Liu [11, 12] proved that transonic shock flows in the diverging section of the nozzle are stable, whereas flows with shock waves in a converging duct are dynamically unstable. For the two dimensional isentropic Euler system, Xin and Yin [18] proved the transonic shock is dynamically stable when it is located in the diverging part of nozzles, but dynamically unstable in the converging part of the nozzle. In this paper, we will prove the dynamical stability of the transonic shocks for the full Euler system when the transonic shock lies in the diverging part of the nozzle.

Define a two dimensional nozzle $\mathcal{N}$ with nozzle walls Λ_1 and Λ_2 that are C^4 regular for

$$L_0 \le r = \sqrt{x_1^2 + x_2^2} \le L_0 + 1,$$