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A Note on Cohomological Mirror Symmetry of Toric Manifolds

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Abstract. In this note we describe a logarithmic version of mirror Landau-Ginzburg model for semi-projective toric manifolds and show in an elementary and explicit way that the state space ring of the Landau-Ginzburg mirror is isomorphic to the $\mathbb{C}$-valued cohomology of the toric manifold.

AMS subject classifications: 32Q55

Key words: Mirror symmetry, Landau-Ginzburg model, toric manifold.

1 Introduction

In mirror symmetry beyond the Calabi-Yau case, the mirror of a toric manifold X_Σ is given by a Landau-Ginzburg model (Y,f), where $Y=(\mathbb{C}^*)^n$ and f is a Laurent polynomial defined on Y. The corresponding mirror symmetry theorems are well established during the past years, see for example [3, 5, 10, 14]. Among various results, the isomorphism between the A-model moduli space of X and the B-model moduli space of its mirror (Y,f) is the fundamental one. This isomorphism identifies the symplectic geometric structure of X_Σ near the large volume limit and the complex geometry structure of (Y,f) near the large complex structure limit. In particular, it implies the quantum cohomology ring of X_Σ is isomorphic to the Jacobi ring of (Y,f).

In this note, we concern the cohomologies at the limit points of the moduli spaces, that is, we will compare the ordinary cohomology ring of a toric manifold

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X_Σ with the state space ring of a limiting version of its mirror. Motivated by the work of Gross and Siebert, we define such mirror to be a logarithmic version of Landau-Ginzburg model $(Y_\Sigma^\dagger, f)$, where $Y_\Sigma^\dagger$ is a log Calabi-Yau variety with a log smooth map to the standard log point $0^\dagger$, and f is a holomorphic function on $Y_\Sigma^\dagger$. For a fixed semi-projective toric manifold X_Σ, the pair $(Y_\Sigma^\dagger, f)$ can be defined combinatorially from the defining fan Σ. In fact, such $(Y_\Sigma^\dagger, f)$ is the $q\to 0$ limit of the Hori-Vafa mirror [9]. The log structure and the log smooth map are needed to record the limiting process.

The idea is as follows. Given a general toric manifold X_Σ with defining fan Σ, it can be viewed as a gluing of affine toric varieties, and its cohomology can be computed by a Čech double complex. On the other hand, we can construct from combinatorial data of the fan Σ an affine space Y_Σ and define a differential graded algebra (DGA) $L(Y_\Sigma)$ on it. The point is that we can view Y_Σ as a gluing of affine spaces and consider the restriction of $L(Y_\Sigma)$ on each affine piece. There is a one-to-one correspondence between the pieces on both sides and corresponding pieces have isomorphic cohomologies. By technically introducing a third DGA, we show that the de Rham DGA $(\mathcal{A}^\bullet(X_\Sigma),\mathrm{d})$ is quasi-isomorphic to $L(Y_\Sigma)$. When X_Σ is semi-projective, Y_Σ can be equipped with a log structure and there is a natural map from $Y_\Sigma^\dagger$ to the standard log point $0^\dagger$. We show that $L(Y_\Sigma)$ is isomorphic to the twisted DGA of the Landau-Ginzburg model $(Y_\Sigma^\dagger, f)$. In this way, we explicitly connect the cohomology ring of X_Σ with the state space ring of its mirror Landau-Ginzburg model.

The cohomology of toric manifolds is well-known. The use of logarithmic geometry in toric mirror symmetry is also not new. In [3, Section 4.4], the authors defined for toric stacks a logarithmic mirror family and studied the logarithmic twisted de Rham complex. In [2], a logarithmic mirror of projective toric manifolds is used to give a perturbative construction of primitive forms. Mirror symmetry phenomenon for general manifolds is usually complicated, see for example the theory of homological mirror symmetry [12], SYZ conjecture [15] and Gross-Siebert program [6,7]. The novelty of the present note is mainly the explicit cohomological mirror symmetry for toric manifolds. Though elementary, it gives an illustrative example of the mirror symmetry phenomenon.

2 Spaces associated to the defining fan of a toric manifold

Any toric variety can be defined by a fan, so we start by specifying the fan structure. Let $N\cong\mathbb{Z}^n$ be a rank n lattice and M be the dual lattice. Define $N_\mathbb{R}:=N\otimes_\mathbb{Z}\mathbb{R}, M_\mathbb{R}:=M\otimes_\mathbb{Z}\mathbb{R}$. Let Σ be a smooth fan in $N_\mathbb{R}$, which implies the