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A Schur-Horn Type Theorem for Symplectic Matrices

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Abstract. In this note, we establish a new version of Schur-Horn type theorem for symplectic matrices. Meanwhile, we establish a necessary and sufficient condition for the equality to hold in the above result.

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1 Introduction

The symbols $\mathbb{R}^{2n\times 2n}$ and I_n refer to the set of $2n\times 2n$ real matrices and the $n\times n$ identity matrix, respectively. The notation $A>0$ will be used to indicate that A is a positive definite matrix. Denote by J the $2n\times 2n$ matrix $J=\begin{pmatrix} 0 & I_n \\ -I_n & 0\end{pmatrix}$, let

$$\mathrm{Sp}(2n)=\{M\in\mathbb{R}^{2n\times 2n}: M^TJM=J\}$$

be the group of real symplectic matrices. Given $x\in\mathbb{R}^m$, we denote by $x^{\uparrow}=(x_1^{\uparrow},x_2^{\uparrow},\ldots,x_m^{\uparrow})$ the vector whose entries are the coordinates of x rearranged in increasing order $x_1^{\uparrow}\le x_2^{\uparrow}\le\cdots\le x_m^{\uparrow}$. If $x,y\in\mathbb{R}^m$, we write $x\ge y$ (respectively, $x>y$) if all the entries of $x-y$ are nonnegative (respectively, positive).

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Let $H\in\mathbb{R}^{2n\times 2n}$ have a block decomposition

$$H=\begin{pmatrix} L & Y \\ C & Z \end{pmatrix},$$

where L,Y,C,Z are $n\times n$ matrices. Let $\widetilde{H}$ be the $n\times n$ matrix whose entries are given by

$$\widetilde{h}_{ij}=\frac{1}{2}\left(c_{ij}^2+l_{ij}^2+y_{ij}^2+z_{ij}^2\right), \tag{1.1}$$

where $C=[c_{ij}]_{i,j=1}^n, L=[l_{ij}]_{i,j=1}^n, Y=[y_{ij}]_{i,j=1}^n, Z=[z_{ij}]_{i,j=1}^n\in\mathbb{R}^{n\times n}$.

Suppose H be a positive-definite matrix, then $\widetilde{H}$ has many nice properties. Huang and Wang [8, Theroem 2.1] showed that $\widetilde{H}$ is a positive-definite matrix, they also presented a matrix inequality for any principal submatrix of $\widetilde{H}$; see [8, Theorem 2.3]. Meanwhile, Huang [6] established a new version of a Schur-Horn type theorem via the matrix $\widetilde{H}$. Many similar results are obtained for symplectic matrix H. Bhatia and Jain [3, Theorem 6] obtained $\widetilde{H}$ is doubly super-stochastic. The following proposition gives a matrix inequality for any submatrix of $\widetilde{H}$.

Proposition 1.1. *Let $H\in\mathrm{Sp}(2n)$ and $\widetilde{H}$ be the $n\times n$ matrix associated with H according to the rule* (1.1)*. Suppose $Y=(y_{ij})_{l\times l}$ be any $l\times l$ submatrix of $\widetilde{H}$, $1\le l\le n$, we have*

$$\max\{k+l-n,0\}\le\sum_{i,j=1}^{l}y_{ij}\le\frac{1}{2}\sum_{i=1}^{n}\left(s_i^2+\frac{1}{s_i^2}\right),$$

where s_i^2 is the eigenvalue value of H^TH.

Proof. The first inequality follows from [2, Theorem 1]. The Euler decomposition theorem [5] says that every symplectic matrix H can be decomposed as

$$H=O_1\begin{pmatrix} \Gamma & \mathbf{O} \\ \mathbf{O} & \Gamma^{-1} \end{pmatrix}O_2,$$

where O_1 and O_2 are symplectic and orthogonal, $\Gamma=\mathrm{diag}(s_1,s_2,\ldots,s_n)$ with

$$s_1\ge s_2\ge\cdots\ge s_n>0.$$

It follows that

$$\sum_{i,j=1}^{l}y_{ij}\le\sum_{i,j=1}^{n}y_{ij}=\frac{1}{2}\mathrm{tr}(H^TH)=\frac{1}{2}\sum_{i=1}^{n}\left(s_i^2+\frac{1}{s_i^2}\right)$$

as required. □