Commun. Math. Res.
doi: 10.4208/cmr.2025-0040

Vol. **x**, No. x, pp. 1-20
x 2025

Some Complexes and Modules Induced by Strongly FP-injective Modules

Donglin He, Yuyan Li and Wujun Pu*

School of Mathematics and Computer Science, Longnan Normal University, Longnan 742500, P.R. China.

Received 20 September 2025; Accepted 17 November 2025

Abstract. Let R be an associative ring with identity. In this paper, we consider generalizations of Gorenstein FP-injective R-modules and FP-injective complexes, give the definitions and characterizations of strongly Gorenstein FP-injective R-modules and strongly FP-injective complexes, which are induced by strongly FP-injective modules. Then we investigate the strongly FP-injective dimension of complexes.

AMS subject classifications: 16E05, 18G35, 18G20

Key words: Cotorsion pairs, strongly Gorenstein FP-injective modules, strongly FP-injective complexes, strongly FP-injective dimension.

1 Introduction

Gorenstein homological theories play an important role in relative algebra. Auslander [12] introduced the notion of G-dimension of finite R-modules over a commutative Noetherian local ring. Auslander and Bridge [1] extended this notion to two sided Noetherian rings. Enochs and Jenda [3] defined Gorenstein projective modules (not necessarily finitely generated) and Gorenstein injective modules over arbitrary ring. Later, many authors have studied and generalized these notions successively.

*Corresponding author. *Email addresses:* hedonglin@lntc.edu.cn (D. He), nwnulyy@126.com (Y. Li), puwujun@lntc.edu.cn (W. Pu)

In 2013, as an extension of the notion of Gorenstein injective modules, Hu and Zhang [8] introduced the notion of Gorenstein FP-injective R-modules over a left coherent ring based on the complete hereditary cotorsion pair $(\mathcal{FP},\mathcal{FI})$ in R-Mod. Here, the symbol $\mathcal{FP}$ (resp., $\mathcal{FI}$) stands for the subcategory of all FP-projective (resp., FI-injective) modules. Li *et al.* [9] give the definition of strongly FP-injective modules, and proved that $({}^{\perp_1}\mathcal{SFI},\mathcal{SFI})$ is a complete hereditary cotorsion pair over arbitrary ring, where $\mathcal{SFI}$ denotes the subcategory of all strongly FP-injective modules. It is natural to consider another extension of Gorenstein injective modules based on the cotorsion pair $({}^{\perp_1}\mathcal{SFI},\mathcal{SFI})$, that is called strongly Gorenstein FP-injective modules in this paper.

Rozas [11] systematically introduced projective complexes, injective complexes and flat complexes. Yang and Liu [16] give the definition of FP-injective complexes. Wang and Liu [17] further investigated FP-injective dimension of complexes. These works are based on a left coherent ring. One purpose of this paper is to extend above works: we introduce and study strongly FP-injective complexes and strongly FP-injective dimension of complexes over an arbitrary ring.

We now state the main results of this paper.

Theorem 1.1. *Let M be an R-module. Then the following statements are equivalent:*

(1) *M is a strongly Gorenstein* FP*-injective module.*

(2) $M\in({}^{\perp_1}\mathcal{SFI}\cap\mathcal{SFI})^{\perp}$ *and there is an exact sequence* $\cdots\to N_1\to N_0\to M\to 0$ *with each* $N_i\in{}^{\perp_1}\mathcal{SFI}\cap\mathcal{SFI}$*, which is exact under* $\mathrm{Hom}_R({}^{\perp_1}\mathcal{SFI}\cap\mathcal{SFI},-)$.

(3) *There is an exact sequence* $\mathrm{S}=\cdots\to S_1\to S_0\to S_{-1}\to S_{-2}\to\dots$ *with each* $S_i\in\mathcal{SFI}$ *such that* $M\cong Im(S_0\to S_{-1})$*, which is exact under* $\mathrm{Hom}_R(\mathcal{SGFI},-)$.

(4) *There is an exact sequence* $\mathrm{S}=\cdots\to S_1\to S_0\to S_{-1}\to S_{-2}\to\dots$ *with each* $S_i\in\mathcal{SFI}$ *such that* $M\cong Im(S_0\to S_{-1})$*, which is exact under* $\mathrm{Hom}_R({}^{\perp_1}\mathcal{SFI}\cap\mathcal{SFI},-)$.

Theorem 1.2. *Let* X *be a complex. Then the following conditions are equivalent:*

(1) X *is a flat complex.*

(2) $\mathrm{X}^+=\underline{\mathrm{Hom}}(\mathrm{X},D^1(\mathbb{Q}/\mathbb{Z}))$ *is a strongly* FP*-injective complex.*

Theorem 1.3. *Assume that* X *is a strongly* FP*-injective complex and* F *is a finitely presented complex. Then* $\mathrm{Ext}^{i\geqslant 1}(\mathrm{F},\mathrm{X})=0$.

Theorem 1.4. *Let* C *be a complex and n an integer. Then the following assertions about strongly* FP*-injective dimension of* C *are equivalent:*

(1) $\widetilde{\mathcal{SFI}}\text{-id}(\mathrm{C})\leqslant n$.

(2) $\inf H(\mathrm{C}) \geqslant -n$ *and* $\mathrm{Z}_{-n}(\mathrm{I}) \in \mathcal{SFI}$ *for each* dg*-injective resolution* $\mathrm{C} \to \mathrm{I}$.

(3) $\inf H(\mathrm{C}) \geqslant -n$ *and* $\mathrm{Z}_j(\mathrm{I}) \in \mathcal{SFI}$ *for every* $j \leqslant -n$ *for each* dg*-injective resolution* $\mathrm{C} \to \mathrm{I}$.

(4) *There exists a* dg*-injective resolution* $\mathrm{C} \to \mathrm{I}'$ *such that* $H_j(\mathrm{I}') = 0$ *for every* $j \leqslant -n-1$ *and* $\mathrm{Z}_{-n}(\mathrm{I}') \in \mathcal{SFI}$.

(5) *There exists a* dg*-injective resolution* $\mathrm{C} \to \mathrm{I}'$ *such that* $H_j(\mathrm{I}') = 0$ *for every* $j \leqslant -n-1$ *and* $\mathrm{Z}_j(\mathrm{I}') \in \mathcal{SFI}$ *for each* $j \leqslant -n$.

The contents of the paper are summarized as follows. In Section 2, we collect some known notions and results. In Section 3, we introduce strongly Gorenstein FP-injective modules, then give some properties and characterizations of strongly Gorenstein FP-injective modules. Section 4 is devoted to strongly FP-injective complexes and strongly FP-injective dimension of complexes.

2 Preliminaries

Throughout this paper, R denotes an associative ring with identity, R-Mod denotes the category of all left R-modules. By a "module" we always mean a left R-module. By the term "subcategory" we always mean a full additive subcategory closed under isomorphisms.

2.1 Cotorsion pairs

Let $\mathcal{A}$ be an abelian category and $\mathcal{X}$ a subcategory of $\mathcal{A}$. Define

$$
\begin{aligned}
{}^{\perp_1}\mathcal{X} &= \left\{ M \in \mathcal{A} \mid \mathrm{Ext}^1_{\mathcal{A}}(M,X) = 0 \ \text{ for any object } \ X \in \mathcal{X} \right\}, \\
{}^{\perp}\mathcal{X} &= \left\{ M \in \mathcal{A} \mid \mathrm{Ext}^{i \geqslant 1}_{\mathcal{A}}(M,X) = 0 \ \text{ for any object } \ X \in \mathcal{X} \right\}.
\end{aligned}
$$

$\mathcal{X}^{\perp}$ and $\mathcal{X}^{\perp_1}$ are similarly defined. A pair $(\mathcal{X},\mathcal{Y})$ of subcategories of $\mathcal{A}$ is said to be a cotorsion pair if $\mathcal{X}^{\perp_1} = \mathcal{Y}$ and ${}^{\perp_1}\mathcal{Y} = \mathcal{X}$. The cotorsion pair $(\mathcal{X},\mathcal{Y})$ is said to be hereditary if $\mathrm{Ext}^{i \geqslant 1}_{\mathcal{A}}(X,Y) = 0$ for all objects $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. A morphism $\varphi: M \to X$ with $X \in \mathcal{X}$ is called an $\mathcal{X}$-preenvelope of M if for any morphism $f: M \to X'$ with $X' \in \mathcal{X}$, there is a morphism $g: X \to X'$ such that $g\varphi = f$. A monomorphism $\varphi: M \to B$ with $B \in \mathcal{X}$ is said to be a special $\mathcal{X}$ of M if φ is an $\mathcal{X}$-preenvelope of M and $\mathrm{coker} f \in {}^{\perp_1}\mathcal{X}$. Dually, an $\mathcal{X}$-precover and special $\mathcal{X}$-precover are similarly defined. A cotorsion pair is said to be complete provided that every object of $\mathcal{A}$ has a special $\mathcal{Y}$-preenvelope and special $\mathcal{X}$-precover (for more details see [4]).

2.2 FP-injective modules and strongly FP-injective modules

A module M is called FP-injective [14] if $\mathrm{Ext}_R^1(N,M)=0$ for each finitely presented module N. FP-injective modules act in ways similar to injective modules. Let $\mathcal{FI}$ denote the subcategory of all FP-injective modules, and $\mathcal{FP}$ denote the collection of modules N such that $\mathrm{Ext}_R^1(N,M)=0$ for any FP-injective module M, that is called FP-projective module [10]. $(\mathcal{FP},\mathcal{FI})$ forms a complete hereditary cotorsion pair over a left coherent ring. A module M is called a strongly FP-injective module [9], if $\mathrm{Ext}_R^{i\geqslant 1}(N,M)=0$ for each finitely presented module N. Our study of strongly Gorenstein FP-injective modules is related to the coherence criteria obtained via strongly Gorenstein FP-injective modules in [2], which provide ring-theoretic contexts where our notions naturally arise. In what follows, we denote by $\mathcal{SFI}$ the subcategory of all strongly FP-injective modules. It is obvious that $\mathcal{I}\subseteq\mathcal{SFI}\subseteq\mathcal{FI}$, where $\mathcal{I}$ denote the subcategory of all injective modules. Some results in [9] are spread out as follows, which will be used repeatedly in the paper.

Lemma 2.1. *The following statements are hold:*

(1) $\mathcal{SFI}$ *is closed under extensions, products and cokernels of monomorphisms.*

(2) A right R-module M is flat if and only if $M^+=\mathrm{Hom}_R(M,\mathbb{Q}/\mathbb{Z})$ *is a strongly* FP*-injective left R-module.*

(3) $({}^{\perp_1}\mathcal{SFI},\mathcal{SFI})$ *is a complete hereditary cotorsion pair.*

Lemma 2.2. *The following conditions are equivalent:*

(1) R is a left coherent ring.

(2) $\mathcal{SFI}=\mathcal{FI}$.

(3) Every direct limit of strongly FP*-injective left R-modules is* FP*-injective.*

Lemma 2.3. *Let* $0\to U\to V\to W\to 0$ *be an exact sequence of R-Mod, and U a strongly* FP*-injective module. Then V is a strongly* FP*-injective module if and only if W is a strongly* FP*-injective module.*

2.3 Complexes

Let $\mathcal{C}(R)$ denote the abelian category of complexes of left R-modules. A complex of modules

$$\cdots\longrightarrow X_{n+1}\xrightarrow{\delta_{n+1}^X}X_n\xrightarrow{\delta_n^X}X_{n-1}\xrightarrow{\delta_{n-1}^X}\cdots$$

is denoted by X. The n-th cycle (resp., homology) module is defined as Kerδ_n^X (resp., Kerδ_n^X/Imδ_{n+1}^X) and denoted by $Z_n(X)$ (resp., $H_n(X)$). supX=sup$\{i|X_i\neq 0\}$, infX=inf$\{i|X_i\neq 0\}$. Given a module M, we use $D^n(M)$ to denote the complex $\cdots\to 0\to M\to^1 M\to 0\to\cdots$ with M in n-th and $(n-1)$-th positions. We also use $S^n(M)$ to denote the complex with M in degree zero and 0 elsewhere. Let $\mathcal{W}$ be a subcategory of R-Mod. Then a complex X is said to be exact under $\mathrm{Hom}_R(-,\mathcal{W})$ if the complex $\mathrm{Hom}_R(X,W)$ is exact for each $W\in\mathcal{W}$.

A morphism f:X$\to$Y of complexes is a family of homomorphisms $f=(f_n:X_n\to Y_n)_{n\in\mathbb{Z}}$ of modules satisfying $\delta_n^Y f_n=f_{n-1}\delta_n^X$ for all $n\in\mathbb{Z}$. A quasi-isomorphism $f:X\to Y$ is a morphism such that the induced map $H(f):H(X)\to H(Y)$ is an isomorphism for all $n\in\mathbb{Z}$.

For complexes X and Y, denote by $\mathcal{H}$om(X,Y) the complex of $\mathbb{Z}$-modules with n-th component $\mathcal{H}\mathrm{om}(X,Y)_n=\prod_{t\in\mathbb{Z}}\mathrm{Hom}(X_t,Y_{n+t})$ and differential

$$\delta_n(f)=\left(\delta_{n+m}^Y f_m-(-1)^n f_{m-1}\delta_m^X\right)_{m\in\mathbb{Z}}$$

for any $f\in\mathcal{H}\mathrm{om}(X,Y)_n$. It is easy to see that $\mathrm{Hom}(X,Y)=Z_0(\mathcal{H}\mathrm{om}(X,Y))$. Let $\underline{\mathrm{Hom}}(X,Y)=Z(\mathcal{H}\mathrm{om}(X,Y))$. One checks that $\underline{\mathrm{Hom}}(X,Y)$ can be made into a complex in which $\underline{\mathrm{Hom}}(X,Y)_n$ is the abelian group of morphisms from X to $\sum^{-n}Y$ and whose boundary operator is given by $\delta_n(f):X\to\sum^{-(n-1)}Y$ where $\delta_n(f)_m=(-1)^n\delta^Y f_m$ for all $m\in\mathbb{Z}$ and $f\in\underline{\mathrm{Hom}}(X,Y)_n$, in which $\sum^{-n}Y$ is a complex satisfying the condition that $(\sum^{-n}Y)_i=Y_{i+n}$ and whose boundary operators are $(-1)^{-n}\delta_{i+n}^Y$ (for more details see [17]).

Following [5], a complex X is called finitely generated if, in the case where we can write $X=\sum_{i\in I}Y_i$ with $Y_i\in\mathcal{C}(R)$ subcomplexes of X, there exists a finite subset $J\subseteq I$ such that $X=\sum_{j\in J}Y_j$. A complex X is called finitely presented if X is finitely generated and for every exact sequence of complexes $0\to K\to L\to X\to 0$ with L finitely generated, K is also finitely generated.

Lemma 2.4 ([5]). *The following statements are hold:*

(1) A complex X is finitely presented if and only if X is bounded and X_n is finitely presented in R-Mod for all $n\in\mathbb{Z}$.

(2) If M is a finitely presented module, then $S^i(M)$ and $D^i(M)$ are finitely presented complexes for each $i\in\mathbb{Z}$.

Definition 2.1 ([17]). *A complex X is called FP-injective if $\mathrm{Ext}^1(F,X)=0$ for all every finitely presented complex F.*

Lemma 2.5 ([17]). *Let X be a complex. Then the following statements are equivalent:*

(1) X *is an* FP-*injective complex.*

(2) X *is exact and* $Z_n(\mathrm{X})$ *is an* FP-*injective module for all* $n\in\mathbb{Z}$.

(3) X_n *is an* FP-*injective module for all* $n\in\mathbb{Z}$ *and* $\mathcal{H}\mathrm{om}(\mathrm{F},\mathrm{X})$ *is exact for each finitely presented complex* F.

Definition 2.2 ([8]). *Let M be an R-module. M is called a Gorenstein* FP-*injective module if there exists an exact sequence*

$$\mathrm{Y}=\cdots\to Y_2\to Y_1\to Y_0\to Y_{-1}\to Y_{-2}\to\cdots$$

with $Y_i\in\mathcal{FI}$ when $i\geqslant 0$ and Y_i is injective when $i<0$ such that $M\cong Im(Y_0\to Y_{-1})$, and which remains exact whenever $\mathrm{Hom}_R(H,-)$ is applied for any $H\in\mathcal{FP}\cap\mathcal{FI}$. The subcategory of all Gorenstein FP-*injective modules is denoted by $\mathcal{GFI}$.*

3 Strongly Gorenstein FP-injective modules

We start this section with the following definition.

Definition 3.1. *Let M be an R-module. Then M is said to be a strongly Gorenstein* FP-*injective module if there exists an exact sequence*

$$\mathrm{X}=\cdots\to \mathrm{X}_2\xrightarrow{\delta_2^X}\mathrm{X}_1\xrightarrow{\delta_1^X}\mathrm{X}_0\xrightarrow{\delta_0^X}\mathrm{X}_{-1}\xrightarrow{\delta_{-1}^X}\mathrm{X}_{-2}\to\cdots$$

with $\mathrm{X}_i\in\mathcal{SFI}$ when $i\geqslant 0$ and X_i is injective when $i<0$ such that $M\cong Im\delta_0^X$, and X *stays exact under $\mathrm{Hom}_R(S,-)$ for any $S\in{}^{\perp_1}\mathcal{SFI}\cap\mathcal{SFI}$. This exact sequence is called a complete $\mathcal{SFI}$-$\mathcal{I}$ resolution. The subcategory of all strongly Gorenstein* FP-*injective modules is denoted by $\mathcal{SGFI}$.*

Remark 3.1. It is obvious that $\mathrm{Im}\delta_i^X$ is a strongly Gorenstein FP-injective module for each $i\leqslant 0$ in the above definition.

Lemma 3.1. *Let M be a module. Then:*

(1) M is strongly Gorenstein FP-*injective if and only if $M\in({}^{\perp_1}\mathcal{SFI}\cap\mathcal{SFI})^{\perp}$ and there exists an exact sequence $\cdots\to S_1\to S_0\to M\to 0$ with each $S_i\in\mathcal{SFI}$, which is exact under $\mathrm{Hom}_R({}^{\perp_1}\mathcal{SFI}\cap\mathcal{SFI},-)$.*

(2) If M is a strongly FP-*injective module, then $M\in\mathcal{SGFI}$.*

(3) If M is an injective module, then $M\in\mathcal{SGFI}$.

(4) *For each exact sequence* $0 \to U \to V \to W \to 0$ *with* $U \in \mathcal{SFI}$ *and* $W \in \mathcal{SGFI}$, *then* $V \in \mathcal{SGFI}$.

Proof. (1) ($\Rightarrow$) Let M be a strongly Gorenstein FP-injective module. Then there exists an exact sequence

$$\mathrm{X} = \cdots \to X_2 \stackrel{\delta_2^X}{\to} X_1 \stackrel{\delta_1^X}{\to} X_0 \stackrel{\delta_0^X}{\to} X_{-1} \stackrel{\delta_{-1}^X}{\to} X_{-2} \to \cdots$$

with $X_i \in \mathcal{SFI}$ when $i \geqslant 0$ and X_i is injective when $i < 0$ such that $M \cong \mathrm{Im}\delta_0^X$, and X stays exact under $\mathrm{Hom}_R(S,-)$ for any $S \in {}^{\perp_1}\mathcal{SFI} \cap \mathcal{SFI}$. It is sufficient to prove that $M \in ({}^{\perp_1}\mathcal{SFI} \cap \mathcal{SFI})^{\perp}$. Consider the short exact sequence $0 \to M \to X_{-1} \to \mathrm{Im}\delta_{-1}^X \to 0$. By applying the functor $\mathrm{Hom}_R(S,-)$ to the sequence for any $S \in {}^{\perp_1}\mathcal{SFI} \cap \mathcal{SFI}$, we can get an exact sequence $\cdots \to \mathrm{Hom}_R(S,X_{-1}) \to \mathrm{Hom}_R(S,\mathrm{Im}\delta_{-1}^X) \to \mathrm{Ext}_R^1(S,M) \to 0$. Since $\mathrm{Hom}_R(S,X_{-1}) \to \mathrm{Hom}_R(S,\mathrm{Im}\delta_{-1}^X)$ is exact, so $\mathrm{Ext}_R^1(S,M) = 0$. By the shifting formula of dimension and Remark 3.1, it is easy to check that $\mathrm{Ext}_R^{i+1}(S,M) \cong \mathrm{Ext}_R^1(S,\mathrm{Im}\delta_{-i}^X)$ for any $i \geqslant 0$, hence $M \in ({}^{\perp_1}\mathcal{SFI} \cap \mathcal{SFI})^{\perp}$.

($\Leftarrow$) Assume that $M \in ({}^{\perp_1}\mathcal{SFI} \cap \mathcal{SFI})^{\perp}$ and there exists an exact sequence $(\xi) = \cdots \to S_1 \to S_0 \to M \to 0$ with each $S_i \in \mathcal{SFI}$, which is exact under $\mathrm{Hom}_R({}^{\perp_1}\mathcal{SFI} \cap \mathcal{SFI}, -)$. Let

$$(\xi_1) = 0 \to M \to S_{-1} \to S_{-2} \to \cdots$$

be an injective resolution of M with S_i an injective module for each $i \leqslant -1$. Since $M \in ({}^{\perp_1}\mathcal{SFI} \cap \mathcal{SFI})^{\perp}$ and $S_i \in ({}^{\perp_1}\mathcal{SFI} \cap \mathcal{SFI})^{\perp}$ for any $i \leqslant -1$, we can get (ξ_1) is exact under $\mathrm{Hom}_R({}^{\perp_1}\mathcal{SFI} \cap \mathcal{SFI}, -)$ by [13, Lemma 2.9]. Hence, M is strongly Gorenstein FP-injective by combining (ξ) and (ξ_1).

(2) Let M be a strongly FP-injective module. Then $M \in ({}^{\perp_1}\mathcal{SFI})^{\perp} \subseteq ({}^{\perp_1}\mathcal{SFI} \cap \mathcal{SFI})^{\perp}$ by Lemma 2.1(3). Considering the exact sequence

$$(\xi_2) = \cdots \to 0 \to 0 \to M \stackrel{1_M}{\to} M \to 0 \to \cdots,$$

one can check that (ξ_2) is exact under $\mathrm{Hom}_R({}^{\perp_1}\mathcal{SFI} \cap \mathcal{SFI}, -)$. According to (1), we can know $M \in \mathcal{SGFI}$.

(3) It is obvious.

(4) Let $0 \to U \to V \to W \to 0$ be an exact sequence with $U \in \mathcal{SFI}$ and $W \in \mathcal{SGFI}$. Since $W \in \mathcal{SGFI}$, there is an exact sequence $0 \to K \to S \to W \to 0$

with $K\in\mathcal{SGFI}$ and $S\in\mathcal{SFI}$ by the proof of (1), which is exact under $\mathrm{Hom}_R({}^{\perp_1}\mathcal{SFI}\cap\mathcal{SFI},-)$. Construct the pullback diagram (Diagram 1).

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K & = & K & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U & \longrightarrow & D & \longrightarrow & S & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U & \longrightarrow & V & \longrightarrow & W & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Diagram 1: The pullback diagram of $V\to W$ and $S\to W$.

Since $U,S\in\mathcal{SFI}$ and $\mathcal{SFI}$ is closed under extensions by Lemma 2.1, we have $D\in\mathcal{SFI}$. Note that the right column and the middle row are exact under $\mathrm{Hom}_R({}^{\perp_1}\mathcal{SFI}\cap\mathcal{SFI},-)$. It is not difficult to get that the middle column is also exact under $\mathrm{Hom}_R({}^{\perp_1}\mathcal{SFI}\cap\mathcal{SFI},-)$. According to (1), we know $U,W\in({}^{\perp_1}\mathcal{SFI}\cap\mathcal{SFI})^{\perp}$, so $V\in({}^{\perp_1}\mathcal{SFI}\cap\mathcal{SFI})^{\perp}$. Therefore, $V\in\mathcal{SGFI}$ by (1). □

Lemma 3.2. *Let M be a module. Then:*

(1) $\mathcal{SGFI}$ is closed under extensions, cokernels of monomorphisms and summands.

(2) Every kernels in a complete $\mathcal{SFI}$-$\mathcal{I}$ resolution is in $\mathcal{SGFI}$.

(3) Let $0\to U\to V\to W\to 0$ be an exact sequence of R-modules. If $V,W\in\mathcal{SGFI}$, then $U\in\mathcal{SGFI}$ if and only if $\mathrm{Ext}^1_R(N,U)=0$ for any $N\in{}^{\perp_1}\mathcal{SFI}\cap\mathcal{SFI}$.

Proof. In order to prove the conclusion, let us firstly prove that there exists an exact sequence $(\eta)=\cdots\to N_1\to N_0\to M\to 0$ with each $N_i\in{}^{\perp_1}\mathcal{SFI}\cap\mathcal{SFI}$ such that (η) is exact under $\mathrm{Hom}_R({}^{\perp_1}\mathcal{SFI}\cap\mathcal{SFI},-)$ for any $M\in\mathcal{SGFI}$. By Lemma 3.1(1), there is an exact sequence $0\to K_0\to S_0\to M\to 0$ with $S_0\in\mathcal{SFI}$ and $K_0\in\mathcal{SGFI}$. Note that $({}^{\perp_1}\mathcal{SFI},\mathcal{SFI})$ is a complete hereditary cotorsion pair, there is a special ${}^{\perp_1}\mathcal{SFI}$-precover of S_0, $0\to L_0\to N_0\to S_0\to 0$ with $N_0\in{}^{\perp_1}\mathcal{SFI}$ and $L_0\in\mathcal{SFI}$. Since

$\mathcal{SFI}$ is closed under extensions, so $N_0 \in {}^{\perp_1}\mathcal{SFI} \cap \mathcal{SFI}$. Construct the pullback diagram (Diagram 2).

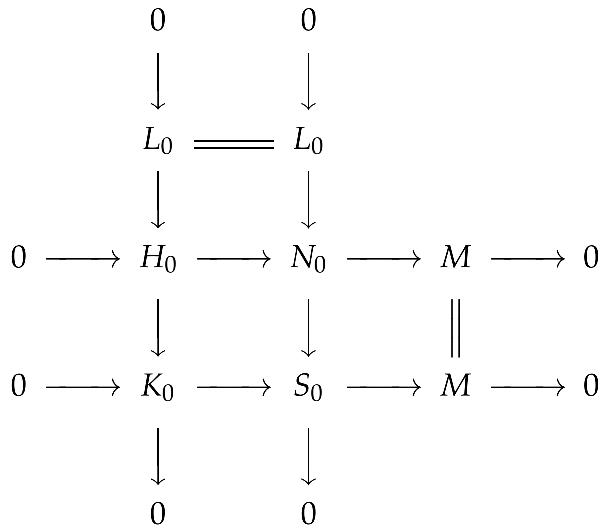

Diagram 2: The pullback diagram of $N_0 \to S_0$ and $K_0 \to S_0$.

Since $L_0 \in \mathcal{SFI}$ and $K_0 \in \mathcal{SGFI}$ in the left column, one can see that $H_0 \in \mathcal{SGFI}$ by Lemma 3.1(4). Because the middle column and the bottom row are exact under $\mathrm{Hom}_R({}^{\perp_1}\mathcal{SFI} \cap \mathcal{SFI}, -)$, we can get $0 \to H_0 \to N_0 \to M \to 0$ is exact under $\mathrm{Hom}_R({}^{\perp_1}\mathcal{SFI} \cap \mathcal{SFI}, -)$. Proceed the process to H_0, and so on, it is not difficult to get an exact sequence $(\eta) = \cdots \to N_1 \to N_0 \to M \to 0$ with each $N_i \in {}^{\perp_1}\mathcal{SFI} \cap \mathcal{SFI}$ such that (η) is exact under $\mathrm{Hom}_R({}^{\perp_1}\mathcal{SFI} \cap \mathcal{SFI}, -)$ for any $M \in \mathcal{SGFI}$.

Let $0 \to U \to V \to W \to 0$ be an exact sequence of R-modules. If $U, W \in \mathcal{SGFI}$, then there exists exact sequences $(\eta_1) = \cdots \to L_1 \to L_0 \to U \to 0$ and $(\eta_2) = \cdots \to N_1 \to N_0 \to W \to 0$ with each $L_i, N_i \in {}^{\perp_1}\mathcal{SFI} \cap \mathcal{SFI}$ such that (η_1) and (η_2) are exact under $\mathrm{Hom}_R({}^{\perp_1}\mathcal{SFI} \cap \mathcal{SFI}, -)$. Note that $\mathcal{SGFI}$ is closed under arbitrary products by Definition 3.1. According to [4, Lemma 8.2.11], we can obtain an exact sequence $(\eta_3) = \cdots \to D_1 \to D_0 \to M \to 0$ with each $D_i \in {}^{\perp_1}\mathcal{SFI} \cap \mathcal{SFI}$ such that (η_3) is exact under $\mathrm{Hom}_R({}^{\perp_1}\mathcal{SFI} \cap \mathcal{SFI}, -)$. Because $U, W \in \mathcal{SGFI}$ and $({}^{\perp_1}\mathcal{SFI} \cap \mathcal{SFI})^{\perp}$ is closed under extensions, one can see $V \in ({}^{\perp_1}\mathcal{SFI} \cap \mathcal{SFI})^{\perp}$, so $V \in \mathcal{SGFI}$ by Lemma 3.1(1). Hence, $\mathcal{SGFI}$ is closed under extensions.

If $U, V \in \mathcal{SGFI}$, then there is an exact sequence $0 \to K_0 \to S_0 \to V \to 0$ with $K_0 \in \mathcal{SGFI}$ and $S_0 \in \mathcal{SFI}$, which is exact under $\mathrm{Hom}_R({}^{\perp_1}\mathcal{SFI} \cap \mathcal{SFI}, -)$. Construct the pullback diagram (Diagram 3).

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & K_0 & = & K_0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & D & \longrightarrow & S_0 & \longrightarrow & W & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & U & \longrightarrow & V & \longrightarrow & W & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Diagram 3: The pullback diagram of $U \to V$ and $S_0 \to V$.

Because the bottom row and the middle column are exact under $\mathrm{Hom}_R({}^{\perp_1}\mathcal{SFI} \cap \mathcal{SFI},-)$, so is the middle row. Since $K_0, U \in \mathcal{SGFI}$ and $\mathcal{SGFI}$ is closed under extensions, one can get $D \in \mathcal{SGFI}$. So there is an exact sequence $(\eta_4) = \cdots \to H_1 \to H_0 \to D \to 0$ with each $H_i \in {}^{\perp_1}\mathcal{SFI} \cap \mathcal{SFI}$ such that (η_4) is exact under $\mathrm{Hom}_R({}^{\perp_1}\mathcal{SFI} \cap \mathcal{SFI},-)$. Combining (η_4) and $0 \to D \to S_0 \to W \to 0$, one easily check that there is an exact sequence $(\eta_5) = \cdots \to H_1 \to H_0 \to S_0 \to W \to 0$ with each H_i and S_0 in $\mathcal{SFI}$. By Lemma 3.1(1) and $U, V \in \mathcal{SGFI}$, we get $W \in ({}^{\perp_1}\mathcal{SFI} \cap \mathcal{SFI})^{\perp}$. Thus, $W \in \mathcal{SGFI}$. Therefore, $\mathcal{SGFI}$ is closed under cokernels of monomorphisms.

Since $\mathcal{SGFI}$ is closed under extensions and products, it is easy to prove that $\mathcal{SGFI}$ is also closed under summands by [7, Proposition 1.4].

(2) Let

$$\mathrm{X} = \cdots \to X_2 \xrightarrow{\delta_2^X} X_1 \xrightarrow{\delta_1^X} X_0 \xrightarrow{\delta_0^X} X_{-1} \xrightarrow{\delta_{-1}^X} X_{-2} \to \cdots$$

be a complete $\mathcal{SFI}$-$\mathcal{I}$ resolution and $K_n = \mathrm{Ker}\delta_n^X$ for all $n \in \mathbb{Z}$. One can get $\mathrm{Ker}\delta_n^X = \mathrm{Im}\delta_{n+1}^X$ is a strongly Gorenstein $\mathcal{FP}$-injective module for all $n \leqslant -1$ by Remark 3.1. Suppose $n \geqslant 0$, since $X_i \in \mathcal{SFI} \subseteq ({}^{\perp_1}\mathcal{SFI} \cap \mathcal{SFI})^{\perp}$ for each $i \in \mathbb{Z}$ and X stays exact under $\mathrm{Hom}_R({}^{\perp_1}\mathcal{SFI} \cap \mathcal{SFI},-)$, we obtain that

$$\mathrm{Ext}^1(N,K_i) = 0, \quad \mathrm{Ext}_R^j(N,K_n) \cong \mathrm{Ext}_R^1(N,K_{n+j-1})$$

for any $N \in {}^{\perp_1}\mathcal{SFI} \cap \mathcal{SFI}$ and any $j \geqslant 1$. So $\mathrm{Ext}_R^{j\geqslant 1}(N,K_n) = 0$, which means $K_n \in$

$({}^{\perp_1}\mathcal{SFI}\cap\mathcal{SFI})^{\perp}$. By Lemma 3.1(1), one get $K_n\in\mathcal{SGFI}$ for any $n\geqslant 0$. Therefore, every kernel in any complete $\mathcal{SFI}$-$\mathcal{I}$ resolution is in $\mathcal{SGFI}$.

(3) ($\Rightarrow$) It is obvious by Lemma 3.1(1).

($\Leftarrow$) Assume that $\mathrm{Ext}^1(F,U)=0$ for any $F\in{}^{\perp_1}\mathcal{SFI}\cap\mathcal{SFI}$. Since $W\in\mathcal{SGFI}$, there is an exact sequence $0\to K\to N\to W\to 0$ with $K\in\mathcal{SGFI}$ and $N\in{}^{\perp_1}\mathcal{SFI}\cap\mathcal{SFI}$ by the proof of (1) such that it is exact under $\mathrm{Hom}_R({}^{\perp_1}\mathcal{SFI}\cap\mathcal{SFI},-)$. Construct the pullback diagram (Diagram 4).

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K & = & K & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U & \longrightarrow & B & \longrightarrow & N & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U & \longrightarrow & V & \longrightarrow & W & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Diagram 4: The pullback diagram of $V\to W$ and $N\to W$.

Since $K,V\in\mathcal{SGFI}$, so $B\in\mathcal{SGFI}$ by (1). By assumption $\mathrm{Ext}^1(N,U)=0$, one know U is a direct summand of B. Hence, $U\in\mathcal{SGFI}$ by (1). □

Theorem 3.1. *Let M be a module. Then the following statements are equivalent:*

(1) M is a strongly Gorenstein FP*-injective module.*

(2) $M\in({}^{\perp_1}\mathcal{SFI}\cap\mathcal{SFI})^{\perp}$ and there is an exact sequence $\cdots\to S_1\to S_0\to M\to 0$ with each $S_i\in\mathcal{SFI}$, which is exact under $\mathrm{Hom}_R({}^{\perp_1}\mathcal{SFI}\cap\mathcal{SFI},-)$.

(3) There is an exact sequence $\cdots\to S_1\to S_0\to S_{-1}\to S_{-2}\to\cdots$ with each $S_i\in\mathcal{SGFI}$ such that $M\cong\mathrm{Im}(S_0\to S_{-1})$ which is exact under $\mathrm{Hom}_R(\mathcal{SGFI},-)$.

(4) There is an exact sequence $\cdots\to S_1\to S_0\to S_{-1}\to S_{-2}\to\cdots$ with each $S_i\in\mathcal{SGFI}$ such that $M\cong\mathrm{Im}(S_0\to S_{-1})$ which is exact under $\mathrm{Hom}_R({}^{\perp_1}\mathcal{SFI}\cap\mathcal{SFI},-)$.

Proof. (1)⇔(2) It is easy to prove it by Lemma 3.1. (1)⇒(3), (2)⇒(4), and (3)⇒(4) are trivial.

(4)⇒(1) Assume that there is an exact sequence

$$\mathrm{S}=\cdots\to S_1\to S_0\to S_{-1}\to S_{-2}\to\cdots$$

with each $S_i\in\mathcal{SGFI}$ such that $M\cong\mathrm{Im}(S_0\to S_{-1})$ which is exact under $\mathrm{Hom}_R({}^{\perp_1}\mathcal{SFI}\cap\mathcal{SFI},-)$. By Lemma 3.1(1) and $S_i\in\mathcal{SGFI}$, we get $S_i\in({}^{\perp_1}\mathcal{SFI}\cap\mathcal{SFI})^{\perp}$ for any $i\in\mathbb{Z}$. Meanwhile, S is exact under $\mathrm{Hom}_R({}^{\perp_1}\mathcal{SFI}\cap\mathcal{SFI},-)$, it is not difficult to obtain that $\mathrm{Ext}^1(N,K_i)=0$ and $\mathrm{Ext}^j_R(N,K_n)\cong\mathrm{Ext}^1_R(N,K_{n+j-1})$ for any $N\in{}^{\perp_1}\mathcal{SFI}\cap\mathcal{SFI}$, any $i,n\in\mathbb{Z}$ and $j\geqslant 1$, where K_n=$\mathrm{Ker}(S_n\to S_{n-1})$. So $\mathrm{Ext}^{j\geqslant 1}_R(N,K_n)=0$, which means $K_n\in({}^{\perp_1}\mathcal{SFI}\cap\mathcal{SFI})^{\perp}$ for any $n\in\mathbb{Z}$. Of course $M\in({}^{\perp_1}\mathcal{SFI}\cap\mathcal{SFI})^{\perp}$. Since $S_0\in\mathcal{SGFI}$, there is an exact sequence $0\to S_0'\to N_0\to S_0\to 0$ with $N_0\in\mathcal{SFI}$ and $S_0'\in\mathcal{SGFI}$, that is exact under $\mathrm{Hom}_R({}^{\perp_1}\mathcal{SFI}\cap\mathcal{SFI},-)$. Construct the pullback diagram (Diagram 5).

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & S_0' & = & S_0' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & M_0 & \longrightarrow & N_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & K_0 & \longrightarrow & S_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Diagram 5: The pullback diagram of $K_0\to S_0$ and $N_0\to S_0$.

Since the middle column and the bottom row of Diagram 5 are exact under $\mathrm{Hom}_R({}^{\perp_1}\mathcal{SFI}\cap\mathcal{SFI},-)$, so is the middle row. Note that $M_0\in\mathcal{SGFI}$ by Lemma 3.2. Construct the pullback diagram (Diagram 6).

One can see that $M_1\in\mathcal{SGFI}$ by Lemma 3.2. Since the bottom row and the right column are exact under $\mathrm{Hom}_R({}^{\perp_1}\mathcal{SFI}\cap\mathcal{SFI},-)$, so is the middle column. Hence, the sequence $\cdots\to S_3\to S_2\to M_1\to M_0\to 0$ is exact under $\mathrm{Hom}_R({}^{\perp_1}\mathcal{SFI}\cap$

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K_1 & = & K_1 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S_0' & \longrightarrow & M_1 & \longrightarrow & S_1 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S_0' & \longrightarrow & M_0 & \longrightarrow & K_0 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Diagram 6: The pullback diagram of $M_0 \to K_0$ and $S_1 \to K_0$.

$\mathcal{SFI},-)$, which is combined by $\cdots \to S_3 \to S_2 \to K_1 \to 0$ and $0 \to K_1 \to M_1 \to M_0 \to 0$. Repeat the process of M to M_0, and so on, we can obtain an exact sequence $\cdots \to N_2 \to N_1 \to N_0 \to M \to 0$ with each $N_i \in \mathcal{SFI}$. Therefore, M is a strongly Gorenstein FP-injective module by Lemma 3.1. □

4 Strongly FP-injective complexes and strongly FP-injective dimension of complexes

In this section, we introduce the concept of strongly FP-injective complexes, and discuss some properties and characterizations of strongly FP-injective complexes in virtue of the cotorsion pair $({}^{\perp_1}\mathcal{SFI}, \mathcal{SFI})$. Finally, we study strongly FP-injective dimensions of complexes.

Definition 4.1. *A complex* X *is called strongly* FP-*injective if* X *is exact and* $Z_n(\mathrm{X})$ *is a strongly* FP-*injective module for all* $n \in \mathbb{Z}$.

Proposition 4.1. *Let* X *be a complex. If* X *is a strongly* FP-*injective complex, then* X *is also an* FP-*injective complex.*

Proof. Assume that X is a strongly FP-injective complex. By Definition 4.1, X is exact and $Z_n(\mathrm{X})$ is a strongly FP-injective module for all $n \in \mathbb{Z}$. Since strongly FP-injective modules are FP-injective, one can know $Z_n(\mathrm{X})$ is a strongly FP-injective module for all $n \in \mathbb{Z}$. By Lemma 2.5, X is also an FP-injective complex. □

Proposition 4.2. *Let M be a module. Then the following conditions are equivalent:*

(1) M is a strongly FP-*injective module.*

(2) $D^i(M)$ is a strongly FP-*injective complex.*

Recall that a complex X is a flat (resp., injective, projective) if and only if X is exact and $Z_n(\mathrm{X})$ is a flat (resp., injective, projective) module for all $n\in\mathbb{Z}$.

Theorem 4.1. *Let* X *be a complex. Then the following statements are equivalent:*

(1) X *is a flat complex.*

(2) $\mathrm{X}^+ = \underline{\mathrm{Hom}}(\mathrm{X}, D^1(\mathbb{Q}/\mathbb{Z}))$ *is a strongly* FP-*injective complex.*

Proof. Assume that $\mathrm{X} = \cdots \to X_{n+1} \to X_n \to X_{n-1} \to \cdots$. Then X^+ can be made into the complex

$$\cdots \to \mathrm{Hom}\left(\mathrm{X}, \sum^{-n-1} D^1(\mathbb{Q}/\mathbb{Z})\right) \to \mathrm{Hom}\left(\mathrm{X}, \sum^{-n} D^1(\mathbb{Q}/\mathbb{Z})\right) \to \mathrm{Hom}\left(\mathrm{X}, \sum^{-n+1} D^1(\mathbb{Q}/\mathbb{Z})\right) \to \cdots$$

Let $\alpha^{-n}: \mathrm{Hom}(\mathrm{X}, \sum^{-n} D^1(\mathbb{Q}/\mathbb{Z})) \to \mathrm{Hom}_R(X_{-n}, \mathbb{Q}/\mathbb{Z})$, $f = (f_m)_{m\in\mathbb{Z}} \to \alpha^{-n}(f) = f_{-n}$ for any $n\in\mathbb{Z}$. It is not difficult to prove that α^{-n} is an isomorphism for any $n\in\mathbb{Z}$, and the following diagram is commutative:

$$\begin{array}{ccccccc}
\mathrm{X}^+ = \cdots \to & \mathrm{Hom}(\mathrm{X}, \sum^{-n-1} D^1(\mathbb{Q}/\mathbb{Z})) & \to & \mathrm{Hom}(\mathrm{X}, \sum^{-n} D^1(\mathbb{Q}/\mathbb{Z})) & \to & \mathrm{Hom}(\mathrm{X}, \sum^{-n+1} D^1(\mathbb{Q}/\mathbb{Z})) & \to \cdots \\
& \downarrow \alpha^{-n-1} & & \downarrow \alpha^{-n} & & \downarrow \alpha^{-n+1} & \\
W = \cdots \to & \mathrm{Hom}_R(X_{-n-1}, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & \mathrm{Hom}_R(X_{-n}, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & \mathrm{Hom}_R(X_{-n+1}, \mathbb{Q}/\mathbb{Z}) & \to \cdots
\end{array}$$

One can get isomorphisms

$$Z_n(\mathrm{X}^+) \cong Z_n(W) \cong \mathrm{Hom}_R(Z_n, \mathbb{Q}/\mathbb{Z}) \cong Z_n(\mathrm{X})^+,$$

when X is exact. By Lemma 2.1, we have $Z_n(\mathrm{X})$ is flat if and only if $Z_n(\mathrm{X})^+$ is strongly FP-injective. So $Z_n(\mathrm{X})$ is flat if and only if $Z_n(\mathrm{X}^+)$ is strongly FP-injective. Note that X is an exact complex if and only if W is an exact complex, if and only if X^+ is exact. Therefore, X is a flat complex if and only if $\mathrm{X}^+ = \underline{\mathrm{Hom}}(\mathrm{X}, D^1(\mathbb{Q}/\mathbb{Z}))$ is a strongly FP-injective complex. □

Here are some characterizations of a left coherent ring.

Corollary 4.1. *Let R be a ring and X a complex. Then the following assertions are equivalent:*

(1) R is a left coherent ring.

(2) X is an FP*-injective complex if and only if X is a strongly* FP*-injective complex.*

(3) X is a strongly FP*-injective complex if and only if* X^+ *is a flat complex.*

(4) X is a strongly FP*-injective complex if and only if* X^{++} *is a strongly* FP*-injective complex.*

(5) X is a flat complex if and only if X^+ *is a flat complex.*

Proof. It follows from Lemma 2.1, Theorem 3.1 and [17, Corollary 2.14]. □

Proposition 4.3. *Let X be a strongly* FP*-injective complex. Then X_n is a strongly* FP*-injective module and* $\mathcal{H}\mathrm{om}(\mathrm{F},\mathrm{X})$ *is exact for any finitely presented complex* F.

Proof. According to Proposition 4.1 and Lemma 2.5 one can get $\mathcal{H}\mathrm{om}(\mathrm{F},\mathrm{X})$ is exact for any finitely presented complex F. It is sufficient to prove that X_n is a strongly FP-injective module. Let

$$\mathrm{X}=\cdots\to X_2\xrightarrow{\delta_2^X}X_1\xrightarrow{\delta_1^X}X_0\xrightarrow{\delta_0^X}X_{-1}\xrightarrow{\delta_{-1}^X}X_{-2}\to\cdots,$$

and $0\to X_n\xrightarrow{f}V\to U\to 0$ be an arbitrary extension of X_n by U, in which U is any finitely presented module. Since $\mathrm{Ker}\pi=\mathrm{Im}\delta_n^X\subseteq\mathrm{Ker}\delta_{n-1}^X$ with $\pi:X_{n-1}\to\mathrm{Coker}\delta_n^X$ is a canonical projective. By the factor lemma, there exists homological morphism $g{:}\mathrm{Coker}\delta_n^X\to X_{n-2}$ such that $g\pi=\delta_{n-1}^X$. Construct the pushout diagram (Diagram 7). So Diagram 8 is commutative.

It is easy to get

$$\mathrm{H}=\cdots\to X_{n+2}\xrightarrow{\delta_{n+2}^X}X_{n+1}\xrightarrow{f\delta_{n+1}^X}V\xrightarrow{\alpha}W\xrightarrow{gh}X_{n-2}\xrightarrow{\delta_{n-2}^X}X_{n-3}\to\cdots$$

is a complex. Therefore, $0\to\mathrm{X}\to\mathrm{H}\to D^n(U)\to 0$ is a short exact sequence of complexes. According to Lemma 2.4 and U is a finitely presented module, $D^n(U)$ is a finitely presented complex. By applying the functor $\mathrm{Hom}(D^n(U),-)$ to above short exact sequence, one can get the following long exact sequence:

$$\begin{aligned}0&\to\mathrm{Hom}(D^n(U),\mathrm{X})\to\mathrm{Hom}(D^n(U),\mathrm{H})\\&\to\mathrm{Hom}(D^n(U),D^n(U))\to\mathrm{Ext}^1(D^n(U),\mathrm{X})\to\cdots.\end{aligned}$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X_n & \xrightarrow{f} & V & \longrightarrow & U & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle \delta_n^X} & & \downarrow{\scriptstyle \alpha} & & \| & & \\
0 & \longrightarrow & X_{n-1} & \xrightarrow{\beta} & W & \longrightarrow & U & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle h} & & & & \\
 & & \mathrm{Coker}\delta_n^X & = & \mathrm{Coker}\delta_n^X & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Diagram 7: The pushout diagram of $X_n \to X_{n-1}$ and $X_n \to V$.

$$\begin{array}{ccccccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & X_{n+2} & \xrightarrow{\delta_{n+2}^X} & X_{n+1} & \xrightarrow{\delta_{n+1}^X} & X_n & \xrightarrow{\delta_n^X} & X_{n-1} & \xrightarrow{\delta_{n-1}^X} & X_{n-2} & \xrightarrow{\delta_{n-2}^X} & X_{n-3} & \longrightarrow & \cdots\\
 & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \\
\cdots & \longrightarrow & X_{n+2} & \xrightarrow{\delta_{n+2}^X} & X_{n+1} & \xrightarrow{f\delta_{n+1}^X} & V & \xrightarrow{\alpha} & W & \xrightarrow{gh} & X_{n-2} & \xrightarrow{\delta_{n-2}^X} & X_{n-3} & \longrightarrow & \cdots\\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & U & \xrightarrow{1} & U & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots\\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

Diagram 8: The joined diagram.

Noting that X is a strongly FP-injective complex, and strongly FP-injective complexes are FP-injective, we know $\mathrm{Ext}^1(D^n(U),\mathrm{X})=0$, which means $0\to \mathrm{X}\to \mathrm{H}\to D^n(U)\to 0$ is split. So the n-th term $0\to X_n\to V\to U\to 0$ is split, and $\mathrm{Ext}_R^{i\geqslant 1}(U,X_n)=0$. Thus, X_n is a strongly FP-injective module. $\square$

Gillespie introduced in [6] several classes of complexes associated to a cotorsion pair in an abelian category. Specializing to the cotorsion pair $({}^{\perp_1}\mathcal{SFI},\mathcal{SFI})$, Gillespie's definition reduces to the following.

Definition 4.2. *Let R be a ring and X a complex.*

(1) X is called a ${}^{\perp_1}\mathcal{SFI}$-complex if it is exact and $\mathrm{Z}_i(\mathrm{X}) \in {}^{\perp_1}\mathcal{SFI}$ for each $i \in \mathbb{Z}$.

(2) X is called an $\mathcal{SFI}$-complex if it is exact and $\mathrm{Z}_i(\mathrm{X}) \in \mathcal{SFI}$ for each $i \in \mathbb{Z}$.

(3) X is called a dg-${}^{\perp_1}\mathcal{SFI}$ complex if $\mathrm{X}_i \in {}^{\perp_1}\mathcal{SFI}$ for each $i \in \mathbb{Z}$, and $\mathcal{H}\mathrm{om}(\mathrm{X},\mathrm{S})$ is exact whenever S is an $\mathcal{SFI}$-complex.

(4) X is called a dg-$\mathcal{SFI}$ complex if $X_i \in \mathcal{SFI}$ for each $i \in \mathbb{Z}$, and $\mathcal{H}\mathrm{om}(\mathrm{T},\mathrm{X})$ is exact whenever T is a ${}^{\perp_1}\mathcal{SFI}$-complex.

Remark 4.1. (1) $\mathcal{SFI}$-complexes are coincide with strongly FP-injective complexes.

(2) The class of all ${}^{\perp_1}\mathcal{SFI}$-complexes (resp., dg-${}^{\perp_1}\mathcal{SFI}$ complexes) is denoted by $\widetilde{{}^{\perp_1}\mathcal{SFI}}$ (resp., dg-$\widetilde{{}^{\perp_1}\mathcal{SFI}}$). Similarly, the class of all $\mathcal{SFI}$-complexes (resp., dg-$\mathcal{SFI}$ complexes) is denoted by $\widetilde{\mathcal{SFI}}$ (resp., dg-$\widetilde{\mathcal{SFI}}$).

Proposition 4.4. *Let X be a complex. Then X admits a special $\widetilde{{}^{\perp_1}\mathcal{SFI}}$-precover and a special $\widetilde{\mathcal{SFI}}$-preenvelope.*

Proof. Since $({}^{\perp_1}\mathcal{SFI},\mathcal{SFI})$ is a complete hereditary cotorsion pair, it is easy to prove that X admits a special $\widetilde{{}^{\perp_1}\mathcal{SFI}}$-precover and a special $\widetilde{\mathcal{SFI}}$-preenvelope. The proof is complete. $\square$

According to [17, Lemma 3.2] and [15, Theorem 3.5], one can get the following conclusion.

Theorem 4.2. *Let R be a ring and X a complex. Then:*

(1) Complexes that are bounded below and have components in ${}^{\perp_1}\mathcal{SFI}$ are dg-${}^{\perp_1}\mathcal{SFI}$ complexes.

(2) Complexes that are bounded above and have components in $\mathcal{SFI}$ are dg-$\mathcal{SFI}$ complexes.

(3) $(\widetilde{{}^{\perp_1}\mathcal{SFI}}$, dg-$\widetilde{\mathcal{SFI}})$ and (dg-$\widetilde{{}^{\perp_1}\mathcal{SFI}}$, $\widetilde{\mathcal{SFI}})$ are complete hereditary cotorsion pairs in $\mathcal{C}(R)$.

(4) dg-$\widetilde{{}^{\perp_1}\mathcal{SFI}} \cap \boldsymbol{\varepsilon} = \widetilde{{}^{\perp_1}\mathcal{SFI}}$ and dg-$\widetilde{\mathcal{SFI}} \cap \boldsymbol{\varepsilon} = \widetilde{\mathcal{SFI}}$, where $\boldsymbol{\varepsilon}$ denotes the class of exact complexes.

Proof. Since $({}^{\perp_1}\mathcal{SFI},\mathcal{SFI})$ is a complete hereditary cotorsion pair, one can get (1) and (3) by [17, Lemma 3.2], and $(\widetilde{{}^{\perp_1}\mathcal{SFI}}, \text{dg-}\widetilde{\mathcal{SFI}})$ and $(\text{dg-}\widetilde{{}^{\perp_1}\mathcal{SFI}}, \widetilde{\mathcal{SFI}})$ are hereditary cotorsion pairs in $\mathcal{C}(R)$. According to [15, Theorem 3.5], it is not difficult to prove that $(\widetilde{{}^{\perp_1}\mathcal{SFI}}, \text{dg-}\widetilde{\mathcal{SFI}})$ and $(\text{dg-}\widetilde{{}^{\perp_1}\mathcal{SFI}}, \widetilde{\mathcal{SFI}})$ are complete. The proof is complete. □

Theorem 4.3. *Assume that* X *is a strongly* FP*-injective complex and* F *is a finitely presented complex. Then* $\mathrm{Ext}^{i\geqslant 1}(\mathrm{F},\mathrm{X})=0$.

Proof. Since F is a finitely presented complex, we have F is bounded and F_n is a finitely presented module for all $n\in\mathbb{Z}$ by Lemma 2.4. According to Definition 4.1, it is easy to know $F_n \in {}^{\perp_1}\mathcal{SFI}$. Hence, F is a dg-${}^{\perp_1}\mathcal{SFI}$ complex by Theorem 4.2(1). Note that $(\widetilde{{}^{\perp_1}\mathcal{SFI}}, \text{dg-}\widetilde{\mathcal{SFI}})$ is a complete hereditary cotorsion pair, so we can get $\mathrm{Ext}^{i\geqslant 1}(\mathrm{F},\mathrm{X})=0$. □

Now let us introduce strongly FP-injective dimensions of complexes.

Definition 4.3. *A morphism* $\mathrm{X}\to\mathrm{D}$ *is called a dg-$\mathcal{SFI}$ resolution of* X, *if* $\mathrm{X}\to\mathrm{D}$ *is a quasi-isomorphism and* D *is a dg-$\mathcal{SFI}$ complex.*

Proposition 4.5. *Let R be a ring and* X *a complex. Then* X *has an injective dg-$\mathcal{SFI}$ resolution.*

Proof. Since dg-injective complexes are dg-$\mathcal{SFI}$ complexes, and X has an injective dg-injective resolution, it is easy to get that X has an injective dg-$\mathcal{SFI}$ resolution. □

Definition 4.4. *Let* X *be a complex and n an integer. The strongly* FP*-injective dimension of* X, $\widetilde{\mathcal{SFI}}\text{-id}(\mathrm{X})$, *is defined as follows:*

- $\widetilde{\mathcal{SFI}}\text{-id}(\mathrm{X})\leqslant n$ *if there is a quasi-isomorphism* $\mathrm{X}\to\mathrm{D}$ *with* D *is a dg-$\mathcal{SFI}$ complex such that* $\inf \mathrm{D}\geqslant -n$ *and* $Z_i(\mathrm{D})\in\mathcal{SFI}$ *for any integer* $i\leqslant -n$.
- *If* $\widetilde{\mathcal{SFI}}\text{-id}(\mathrm{X})\leqslant n$ *but* $\widetilde{\mathcal{SFI}}\text{-id}(\mathrm{X})\leqslant n-1$ *does not hold, then* $\widetilde{\mathcal{SFI}}\text{-}id(\mathrm{X})=n$.
- *If* $\widetilde{\mathcal{SFI}}\text{-id}(\mathrm{X})\leqslant m$ *for any integer m, then* $\widetilde{\mathcal{SFI}}\text{-id}(\mathrm{X})=-\infty$.
- *If* $\widetilde{\mathcal{SFI}}\text{-id}(\mathrm{X})\leqslant m$ *does not hold for any integer m, then* $\widetilde{\mathcal{SFI}}\text{-id}(\mathrm{X})=+\infty$.

Theorem 4.4. *Let* C *be a complex. Then the following statements are equivalent:*

(1) $\widetilde{\mathcal{SFI}}$-id$(\mathrm{C})\leqslant n$.

(2) $\inf H(\mathrm{C})\geqslant -n$ *and* $Z_{-n}(\mathrm{I})\in\mathcal{SFI}$ *for each dg-injective resolution* $\mathrm{C}\to\mathrm{I}$.

(3) $\inf H(\mathrm{C})\geqslant -n$ *and* $Z_j(\mathrm{I})\in\mathcal{SFI}$ *for any integer* $j\leqslant -n$ *for each dg-injective resolution* $\mathrm{C}\to\mathrm{I}$.

(4) There exists a dg-injective resolution $\mathrm{C}\to\mathrm{I}'$ *such that* $H_j(\mathrm{I}')=0$ *for any integer* $j\leqslant -n-1$ *and* $Z_{-n}(\mathrm{I}')\in\mathcal{SFI}$.

(5) There exists a dg-injective resolution $\mathrm{C}\to\mathrm{I}'$ *such that* $H_j(\mathrm{I}')=0$ *for any integer* $j\leqslant -n-1$ *and* $Z_j(\mathrm{I}')\in\mathcal{SFI}$ *for any* $j\leqslant -n$.

Proof. $(1)\Rightarrow(2)$ Assume that $\widetilde{\mathcal{SFI}}$-id$(\mathrm{C})\leqslant n$. Then there is a dg-$\mathcal{SFI}$ resolution $\mathrm{C}\xrightarrow{f}\mathrm{F}$ such that $\inf H(\mathrm{F})\geqslant -n$ and $Z_{-n}(\mathrm{F})\in\mathcal{SFI}$. For each dg-injective resolution $\mathrm{C}\xrightarrow{g}\mathrm{I}$, $\inf H(\mathrm{I})=\inf H(\mathrm{C})=\inf H(\mathrm{F})\geqslant -n$. By Proposition 4.5, we can assume f is injective. Consider the exact sequence

$$0\to\mathrm{C}\to\mathrm{F}\to\mathrm{L}\to 0$$

with L exact. By applying the functor $\mathrm{Hom}(-,\mathrm{I})$ to above sequence, we get the following exact sequence:

$$0\to\mathrm{Hom}(\mathrm{L},\mathrm{I})\to\mathrm{Hom}(\mathrm{F},\mathrm{I})\to\mathrm{Hom}(\mathrm{C},\mathrm{I})\to\mathrm{Ext}^1(\mathrm{L},\mathrm{I})\to 0.$$

Since I is dg-injective and L is exact, so $\mathrm{Ext}^1(\mathrm{L},\mathrm{I})=0$, and thus there exists a morphism of complexes $h:\mathrm{F}\to\mathrm{I}$ such that $hf=g$. Note that f,g are both quasi-isomorphisms, so is h. We can assume that h is injective (if not, let $\mathrm{F}\to\underline{\mathrm{I}}$ be injective with $\underline{\mathrm{I}}$ an injective complex. Then $\mathrm{F}\to\mathrm{I}\oplus\underline{\mathrm{I}}$ is an injective quasi-isomorphism). Consider the short exact sequence $0\to\mathrm{F}\to\mathrm{I}\to\mathrm{W}\to 0$ with W an exact sequence. Since F and I are both dg-$\mathcal{SFI}$ complexes, so is W. According to Theorem 4.2(3), we get W is an $\mathcal{SFI}$-complex. Consider the following short exact sequence of modules:

$$0\to Z_{-n}(\mathrm{F})\to Z_{-n}(\mathrm{I})\to Z_{-n}(\mathrm{W})\to 0$$

with $Z_{-n}(\mathrm{W})$ and $Z_{-n}(\mathrm{F})$ strongly FP-injective modules. Hence, $Z_{-n}(\mathrm{I})\in\mathcal{SFI}$ by Lemma 2.1.

$(2)\Rightarrow(4)$ is obvious.

$(4)\Rightarrow(1)$ Since dg-injective resolutions are dg-$\mathcal{SFI}$ resolutions, it is easy to prove (1) by Definition 4.3.

$(2)\Rightarrow(3)$ and $(4)\Rightarrow(5)$ One can easily prove them by Lemma 2.1.

$(3)\Rightarrow(2)$ and $(5)\Rightarrow(4)$ are clear. □

Acknowledgments

We thank the referees for their time and comments. The paper is supported by the Gansu Natural Science Foundation Project (No. 25JRRK002).

References

[1] M. Auslander and M. Bridger, *Stable Module Theory*, AMS, 1969.

[2] M. Chen, H. Kim, F. Wang, and X. Zhang, *Some characterizations of coherent rings in terms of strongly FP-injective modules*, Commun. Algebra 48(7) (2020), 2857–2871.

[3] E. E. Enochs and O. M. G. Jenda, *Gorenstein injective and projective modules*, Math. Z. 220(1) (1995), 611–633.

[4] E. E. Enochs and O. M. G. Jenda, *Relative Homological Algebra*, De Gruyter, 2011.

[5] E. E. Enochs and J. R. G. Rozas, *Flat covers of complexes*, J. Algebra 210(1) (1998), 86–102.

[6] J. Gillespie, *The flat model structure on Ch(R)*, Trans. Am. Math. Soc. 356(8) (2004), 3369–3390.

[7] H. Holm, *Gorenstein homological dimensions*, J. Pure Appl. Algebra 189(1-3) (2004), 167–193.

[8] J. Hu and D. Zhang, *Weak AB-context for FP-injective modules with respect to semidualizing modules*, J. Algebra Appl. 12(07) (2013), 1350039.

[9] W. Li, J. Guan, and B. Ouyang, *Strongly FP-injective modules*, Commun. Algebra 45(9) (2017), 3816–3824.

[10] L. Mao and N. Ding, *FP-projective dimensions*, Commun. Algebra 33(4) (2005), 1153–1170.

[11] J. R. G. Rozas, *Covers and Envelopes in the Category of Complexes of Modules*, CRC Press, 1999.

[12] P. Samuel and M. Auslander, *Anneaux de Gorenstein, et torsion en algèbre commutative*, Séminaire d'algèbre Commutative, 1967.

[13] S. Sather-Wagstaff, T. Sharif, and D. White, *AB-contexts and stability for Gorenstein flat modules with respect to semidualizing modules*, Algebras Represent. Theory 14(3) (2011), 403–428.

[14] B. Stenström, *Coherent rings and FP-injective modules*, J. Lond. Math. Soc. 2(2) (1970), 323–329.

[15] G. Yang and Z. Liu, *Cotorsion pairs and model structures on Ch(R)*, Proc. Edinb. Math. Soc. 54(3) (2011), 783–797.

[16] X. Yang and Z. Liu, *FP-injective complexes*, Commun. Algebra 38(1) (2009), 131–142.

[17] Z. Wang and Z. Liu, *FP-injective complexes and FP-injective dimension of complexes*, J. Aust. Math. Soc. 91(2) (2011), 163–187.