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Invertible Linear Maps on the General Linear Lie Algebras Preserving Solvability*

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Abstract: Let M_n be the algebra of all $n \times n$ complex matrices and $gl(n, \mathbb{C})$ be the general linear Lie algebra, where $n \geq 2$. An invertible linear map $\phi : gl(n, \mathbb{C}) \to gl(n, \mathbb{C})$ preserves solvability in both directions if both ϕ and ϕ^{-1} map every solvable Lie subalgebra of $gl(n, \mathbb{C})$ to some solvable Lie subalgebra. In this paper we classify the invertible linear maps preserving solvability on $gl(n, \mathbb{C})$ in both directions. As a sequence, such maps coincide with the invertible linear maps preserving commutativity on M_n in both directions.
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1 Introduction

Let $\mathcal{L}$ be a Lie algebra. Recall that the derived Lie algebra $\mathcal{L}^{(1)}$ of $\mathcal{L}$ is the Lie ideal $[\mathcal{L}, \mathcal{L}]$ spanned by all $[x, y]$, $x, y \in \mathcal{L}$. To each Lie algebra $\mathcal{L}$ we associated the derived series:

$$\mathcal{L} \supseteq \mathcal{L}^{(1)} \supseteq \mathcal{L}^{(2)} = (\mathcal{L}^{(1)})^{(1)} \supseteq \cdots .$$

The Lie algebra $\mathcal{L}$ is solvable if there exists a positive integer r such that $\mathcal{L}^{(r)} = \{0\}$. The set of all $n \times n$ complex matrices is denoted by M_n when considered as a set or a linear space or an algebra. If the linear space M_n is equipped with the Lie product

$$[\,\cdot\,,\,\cdot\,] : [A, B] = AB - BA,$$

then it becomes a general linear Lie algebra, denoted by $gl(n, \mathbb{C})$.

A lot of attention has been paid to linear preserver problem, which concerns the characterization of linear maps on matrix spaces or algebras that leave certain functions, subsets, relations, etc., invariant. The earliest paper on linear preserver problem dates back to 1897 (see [1]), and a great deal of effort has been devoted to the study of this type of question since

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then. One may consult the survey papers [2–4] for details. For linear or nonlinear preserver problem concerning linear Lie algebras we refer to the literature [5–12]. The author in [7] characterized the invertible linear maps on simple Lie algebras of linear types preserving zero Lie products. Radjavi and Semrl in [11] characterized the nonlinear maps which preserve solvability in both directions on the general linear Lie algebras and the special linear Lie algebras. In this article we determine the invertible linear maps preserving solvability on $gl(n,\mathbb{C})$ in both directions, where an invertible linear map $\phi : gl(n,\mathbb{C}) \to gl(n,\mathbb{C})$ is said to preserve solvability in both directions if for any solvable Lie algebra $\mathfrak{s} \subseteq gl(n,\mathbb{C})$, both $\phi(\mathfrak{s})$ and $\phi^{-1}(\mathfrak{s})$ are solvable Lie algebras of $gl(n,\mathbb{C})$. Now we state our main theorem:

Theorem 1.1 *Let $\phi : gl(n,\mathbb{C}) \to gl(n,\mathbb{C})$ be an invertible linear map. The following two conditions are equivalent*:

(1) *ϕ preserves solvability in both directions*;

(2) *There exists a non-zero scalar $\mu \in \mathbb{C}$, a linear functional f on $gl(n,\mathbb{C})$ with $f(I) \neq -\mu$ and an invertible matrix $S \in gl(n,\mathbb{C})$ such that either*

$$\phi(X) = \mu SXS^{-1} + f(X)I$$

for every $X \in gl(n,\mathbb{C})$, or

$$\phi(X) = \mu SX^tS^{-1} + f(X)I$$

for every $X \in gl(n,\mathbb{C})$, where X^t denotes the transpose of X.

The above result determines an explicit form of the linear invertible map preserving solvability described in Theorem 1.1 of [11]. In [12], the author proved that any bijective linear commutativity preserving map ϕ on M_n is also one of the above two standard maps. Thus we have the following corollary.

Corollary 1.1 *Let ϕ be an invertible linear map on $gl(n,\mathbb{C})$. Then the following conditions are equivalent*:

(1) *ϕ preserves solvability in both directions*;

(2) *ϕ preserves zero Lie products in both directions.*

Here we specify some notations for later use. We denote by I the identity matrix in $gl(n,\mathbb{C})$ and by E_{ij} the matrix in $gl(n,\mathbb{C})$ whose sole nonzero entry 1 is in the (i,j)-position. Let $\mathbb{C}I$ be the set $\{aI|a \in \mathbb{C}\}$ of all scalar matrices, H the set of all diagonal matrices in $gl(n,\mathbb{C})$, and $\mathbf{n}^+$ (resp., $\mathbf{n}^-$) the set of all strictly upper (resp., low) triangular matrices. Let $\mathcal{D}$ be the set of the diagonalizable matrices. Denote the one-dimensional vector space $\mathbb{C}E_{st}$ by $\mathcal{L}_{st}$ for any pair (s,t), $1 \leq s \neq t \leq n$. And denote $\mathbb{C}^* = \mathbb{C} - \{0\}$.

2 Certain Invertible Linear Maps Preserving Solvability

In this section, we construct certain invertible linear maps preserving solvability in both directions on $gl(n,\mathbb{C})$, which will be used to describe arbitrary invertible linear maps preserving solvability in both directions.

(A) Inner automorphisms:

For any invertible matrix $T \in gl(n, \mathbb{C})$, the map

$$\sigma_T : gl(n, \mathbb{C}) \to gl(n, \mathbb{C}), \qquad X \mapsto T^{-1}XT,$$

is an automorphism of $gl(n, \mathbb{C})$, called an inner automorphism of $gl(n, \mathbb{C})$.

(B) Graph automorphisms:

Let

$$\omega_0 : gl(n, \mathbb{C}) \to gl(n, \mathbb{C}), \qquad X \mapsto -RX^tR,$$

where

$$R = E_{1n} + E_{2,n-1} + \cdots + E_{n-1,2} + E_{n1}.$$

Then ω_0 is an automorphism of $gl(n, \mathbb{C})$. Both 1 and ω_0 are called graph automorphism of $gl(n, \mathbb{C})$.

(C) Scalar multiplication maps:

For any $c \in \mathbb{C}^*$, define

$$\psi_c : gl(n, \mathbb{C}) \to gl(n, \mathbb{C}), \qquad X \mapsto cX.$$

We call ψ_c a scalar multiplication map on $gl(n, \mathbb{C})$. It is obvious that any scalar multiplication map is an invertible linear map preserving solvability in both directions.

(D) Invertible linear maps induced by a linear function on $gl(n, \mathbb{C})$:

Let $f : gl(n, \mathbb{C}) \to \mathbb{C}$ be a linear function such that

$$f(I) \neq -1.$$

It is easy to see that the map

$$\psi_f : gl(n, \mathbb{C}) \to gl(n, \mathbb{C}), \qquad X \mapsto X + f(X)I$$

is an invertible linear map, and its inverse is the linear map ψ_f^{-1} defined by

$$\psi_f^{-1}(X) = X - \frac{f(X)}{1 + f(I)} I.$$

The map ψ_f is called an invertible linear map induced by the linear function f. Since

$$[\psi_f(X), \psi_f(Y)] = [X, Y]$$

for any $X, Y \in gl(n, \mathbb{C})$, ψ_f preserves solvability in both directions.

The following lemma is easy to check.

Lemma 2.1 (1) $\psi_{c'} \cdot \psi_c = \psi_{c'c}$ *for any* $c, c' \in \mathbb{C}^*$;

(2) $\sigma_{T'} \cdot \sigma_T = \sigma_{TT'}$ *for any pair of invertible matrices* $T, T' \in gl(n, \mathbb{C})$;

(3) $\omega_0^2 = 1$.

By Lemma 2.1 we have

$$\psi_c^{-1} = \psi_{c^{-1}}, \qquad \sigma_T^{-1} = \sigma_{T^{-1}}, \qquad \omega_0^{-1} = \omega_0.$$

3 Proof of the Main Theorem

Before proving the main theorem, we recall some results from Theorem 1.1, Proposition 2.4 and the proof of Lemma 2.5 in [11].

Lemma 3.1 *Let ϕ be a bijective map on $gl(n,\mathbb{C})$ preserving solvability in both directions. Then*

$$\phi(\mathcal{D}) = \mathcal{D}$$

and two diagonalizable matrices A and B commute if and only if $\phi(A)$ and $\phi(B)$ commute. Moreover, let $\mathcal{D}_k$ $(k = 1, 2, \cdots, n)$ be the set of all diagonalizable matrices with exactly k distinct eigenvalues. Then we have

$$\phi(\mathcal{D}_k) = \mathcal{D}_k.$$

In particular,

$$\phi(I) = \lambda I$$

for some nonzero $\lambda \in \mathbb{C}^*$.

Proof of Theorem 1.1 First we prove that Theorem 1.1 holds for $n \geq 3$.

For the sufficient direction, it is easy to see that ϕ is an invertible linear map and its inverse is given by

$$\phi^{-1}(X) = \mu^{-1}S^{-1}XS - \frac{f(S^{-1}XS)}{\mu^2 + \mu f(I)}I$$

or

$$\phi^{-1}(X) = \mu^{-1}S^tX^t(S^t)^{-1} - \frac{f(S^tX^t(S^t)^{-1})}{\mu^2 + \mu f(I)}I$$

for any $X \in gl(n,\mathbb{C})$. Since

$$[\phi(X),\ \phi(Y)] = \mu^2 S[X,Y]S^{-1}$$

or

$$[\phi(X),\ \phi(Y)] = -\mu^2 S([X,Y])^t S^{-1}$$

for any $X, Y \in gl(n,\mathbb{C})$, for any solvable Lie subalgebra $\mathfrak{s}$ of $gl(n,\mathbb{C})$, $\phi(\mathfrak{s})$ is a solvable Lie subalgebra of $gl(n,\mathbb{C})$. Similarly, ϕ^{-1} preserves solvability. Thus ϕ is an invertible linear map preserving solvability in both directions.

Now we prove the essential direction of the theorem. Let ϕ be an invertible linear map on $gl(n,\mathbb{C})$ preserving solvability in both directions. First observe that the image (under ϕ) of a solvable subalgebra generated by a subset X of $gl(n,\mathbb{C})$ is precisely the subalgebra generated by $\phi(X)$. We prove the main theorem through the following nine steps.

Step 1. *There exists an invertible matrix $S_1 \in gl(n,\mathbb{C})$ such that*

$$(\sigma_{S_1} \cdot \phi)(H) = H.$$

For a diagonal matrix

$$h_0 = \mathrm{diag}\{1, 2, \ldots, n\} \in H \subseteq \mathcal{D},$$

we have $\phi(h_0) \in \mathcal{D}$ by Lemma 3.1, and so there exists an invertible matrix $S_1 \in gl(n,\mathbb{C})$ such that

$$(\sigma_{S_1} \cdot \phi)(h_0) = S_1^{-1}\phi(h_0)S_1$$

is a diagonal matrix. Denote

$$\phi_1 = \sigma_{S_1} \cdot \phi.$$

Then ϕ_1 is still an invertible linear map on $gl(n, \mathbb{C})$ preserving solvability in both directions. Let

$$\phi_1(h_0) = \mathrm{diag}\{\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_n\}.$$

Since $h_0 \in \mathcal{D}_n$, by Lemma 3.1, $\phi_1(h_0) \in \mathcal{D}_n$, and so

$$\lambda_i \neq \lambda_j$$

for any $i \neq j$. For any $h \in H$,

$$h_0 \cdot h = h \cdot h_0,$$

so by Lemma 3.1,

$$\mathrm{diag}\{\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_n\} \cdot \phi_1(h) = \phi_1(h) \cdot \mathrm{diag}\{\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_n\}. \tag{3.1}$$

By the above equality (3.1), we know that $\phi_1(h)$ is a diagonal matrix. It follows that

$$\phi_1(H) = H.$$

Step 2. *For any pair (s, t), $1 \leq s \neq t \leq n$, there exists some pair (p, q), $1 \leq p \neq q \leq n$, such that*

$$\phi_1(\mathcal{L}_{st}) = \mathcal{L}_{pq}.$$

Consider $\phi_1(H + \mathcal{L}_{st})$. Since $H + \mathcal{L}_{st}$ is an $(n + 1)$-dimensional solvable subalgebra containing H, $\phi_1(H + \mathcal{L}_{st})$ is also an $(n+1)$-dimensional solvable subalgebra containing H. First we prove that

$$\phi_1(H + \mathcal{L}_{st}) = H + \mathcal{L}_{pq}$$

for some pair (p, q), where $1 \leq p \neq q \leq n$. Denote

$$\mathfrak{s} = \phi_1(H + \mathcal{L}_{st}).$$

For any element $x \in \mathfrak{s}$, write it in the form

$$x = h + \sum_{u \neq v} a_{uv} E_{uv},$$

where $h \in H$, $a_{uv} \in \mathbb{C}$. Let

$$h_1 = \mathrm{diag}\{1, 2, 2^2, \cdots, 2^{n-2}, 2^{n-1}\}.$$

Applying $(\mathrm{ad}\ h_1)$ repeatedly on x, we have

$$\sum_{u \neq v} (2^{u-1} - 2^{v-1})^k a_{uv} E_{uv} = (\mathrm{ad}\ h_1)^k(x) \in \mathfrak{s}, \qquad k = 1, 2, \cdots, n(n-1). \tag{3.2}$$

View the above equations (3.2) as a system of linear equations in $n^2 - n$ variants $(2^{u-1} - 2^{v-1})a_{uv}E_{uv}$ for the pairs (u, v) with coefficients $(2^{u-1} - 2^{v-1})^{k-1}$. For any $(u, v) \neq (u', v')$, it is easy to see that

$$2^{u-1} - 2^{v-1} \neq 2^{u'-1} - 2^{v'-1}.$$

So the determinant of coefficients of variants $(2^{u-1} - 2^{v-1})a_{uv}E_{uv}$, being exactly a Vandermonde determinant, takes a nonzero value. So each $(2^{u-1} - 2^{v-1})a_{uv}E_{uv}$ can be written as a linear combination of

$$(\mathrm{ad}\ h_1)(x),\ (\mathrm{ad}\ h_1)^2(x),\ \cdots,\ (\mathrm{ad}\ h_1)^{n^2-n-1}(x),\ (\mathrm{ad}\ h_1)^{n^2-n}(x).$$

Then $(2^{u-1} - 2^{v-1})a_{uv}E_{uv} \in \mathfrak{s}$. For the case $a_{uv} \neq 0$, $E_{uv} \in \mathfrak{s}$. Since $\dim\mathfrak{s} = n + 1$, there exists only one pair (p, q), $1 \leq p \neq q \leq n$, such that $a_{pq} \neq 0$. Thus

$$\phi_1(H + \mathcal{L}_{st}) = H + \mathcal{L}_{pq}.$$

Assume that

$$\phi_1(E_{st}) = h + aE_{pq},$$

where $h \in H$, $a \in \mathbb{C}^*$. We now need to show that $h = 0$. Otherwise, take

$$h' = \mathrm{diag}\{\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_n\} \in H$$

such that

$$\lambda_p - \lambda_q \neq 0$$

and h', h are linearly independent (do exist). Let

$$h'' = \phi_1^{-1}(h') \in H.$$

Since $\mathcal{L}_{st} + \mathbb{C}h''$ is a two-dimensional solvable subalgebra generated by E_{st} and h'', $\phi_1(\mathcal{L}_{st} + \mathbb{C}h'')$ is a two-dimensional solvable subalgebra generated by $aE_{pq} + h$ and h'. By

$$[h', aE_{pq} + h] = a(\lambda_p - \lambda_q)E_{pq},$$

we see that $h = 0$. Thus

$$\phi_1(E_{st}) = aE_{pq} \in \mathcal{L}_{pq}.$$

Step 3. *There exists some invertible matrix* S_2 *such that*

(1) $(\phi_1 \cdot \sigma_{S_2})(H) = H$;

(2) $(\phi_1 \cdot \sigma_{S_2})(\mathcal{L}_{st}) \subseteq \mathbf{n}^+$ *for any* $1 \leq s < t \leq n$;

(3) $(\phi_1 \cdot \sigma_{S_2})(\mathcal{L}_{st}) \subseteq \mathbf{n}^-$ *for any* $1 \leq t < s \leq n$.

It is not difficult to see that (2) is equivalent to the following announcement:

$$(*) \qquad (\phi_1 \cdot \sigma_{S_2})(\mathcal{L}_{s,s+1}) \subseteq \mathbf{n}^+$$

for any $1 \leq s \leq n-1$.

Since (3) follows from (2) by Step 2, we only need to prove (1) and $(*)$.

Let

$$\mathcal{P}_{\phi_1}^+ = \{(s,t) \mid 1 \leq s < t \leq n,\ \phi_1(\mathcal{L}_{st}) \subseteq \mathbf{n}^+\}.$$

Now we use decreasing induction on Card $\mathcal{P}_{\phi_1}^+$ to complete (1) and $(*)$. If

$$\mathrm{Card}\ \mathcal{P}_{\phi_1}^+ = \frac{n^2-n}{2},$$

i.e., $\phi_1(\mathcal{L}_{st}) \subseteq \mathbf{n}^+$ for any $1 \leq s < t \leq n$, then we choose $S_2 = I$ to complete the proof. If

$$\mathrm{Card}\ \mathcal{P}_{\phi_1}^+ = k < \frac{n^2-n}{2},$$

then there exists at least one $i \in \{1, 2, \cdots, n-1\}$ such that $\phi_1(\mathcal{L}_{i,i+1}) \subseteq \mathbf{n}^-$. Choose an invertible matrix

$$S' = (I - E_{i,i+1})(I + E_{i+1,i})(I - E_{i,i+1}).$$

By an easy computation, we have the following results:

(i) $\sigma_{S'}(\mathrm{diag}\{a_1,\ \cdots,\ a_i,\ a_{i+1},\ \cdots,\ a_n\}) = \mathrm{diag}\{a_1,\ \cdots,\ a_{i+1},\ a_i,\ \cdots,\ a_n\}$, and so

$$\sigma_{S'}(H) = H;$$

(ii) $\sigma_{S'}(E_{i,i+1}) = -E_{i+1,i}$, and so

$$\sigma_{S'}(\mathcal{L}_{i,i+1}) = \mathcal{L}_{i+1,i};$$

(iii) For any $t > i+1$,

$$\sigma_{S'}(E_{it}) = -E_{i+1,t}, \qquad \sigma_{S'}(E_{i+1,t}) = E_{it},$$

and then

$$\sigma_{S'}(\mathcal{L}_{it}) = \mathcal{L}_{i+1,t}, \qquad \sigma_{S'}(\mathcal{L}_{i+1,t}) = \mathcal{L}_{it};$$

(iv) For any $t < i$,

$$\sigma_{S'}(E_{t,i+1}) = -E_{ti}, \qquad \sigma_{S'}(E_{ti}) = E_{t,i+1},$$

and then

$$\sigma_{S'}(\mathcal{L}_{t,i+1}) = \mathcal{L}_{ti}, \qquad \sigma_{S'}(\mathcal{L}_{ti}) = \mathcal{L}_{t,i+1};$$

(v) For any $s \neq i, i+1$ and $t \neq i, i+1$, $1 \leq s < t \leq n$,

$$\sigma_{S'}(E_{st}) = E_{st},$$

and then

$$\sigma_{S'}(\mathcal{L}_{st}) = \mathcal{L}_{st}.$$

Thus we have

(i)

$$(\phi_1 \cdot \sigma_{S'})(H) = H;$$

(ii)

$$(\phi_1 \cdot \sigma_{S'})(\mathcal{L}_{i,i+1}) = \phi_1(\mathcal{L}_{i+1,i}) \subseteq \mathbf{n}^+$$

(if $\phi_1(\mathcal{L}_{i+1,i}) \subseteq \mathbf{n}^-$, then $\phi_1(\mathcal{L}_{i,i+1} + H + \mathcal{L}_{i+1,i}) = H + \phi_1(\mathcal{L}_{i,i+1}) + \phi_1(\mathcal{L}_{i+1,i}) \subseteq H + \mathbf{n}^-$ is solvable, which contradicts the fact that $\mathcal{L}_{i,i+1} + H + \mathcal{L}_{i+1,i}$ is not solvable);

(iii) $\phi_1 \cdot \sigma_{S'}$ induces a permutation on the set $\{\mathcal{L}_{st} | (s,t) \neq (i, i+1),\ 1 \leq s < t \leq n\}$. One will see that the number of pairs (s,t), $1 \leq s < t \leq n$, satisfying that

$$(\phi_1 \cdot \sigma_{S'})(\mathcal{L}_{st}) \subseteq \mathbf{n}^+$$

is precisely $k+1$. By induction hypotheses, there exists an invertible matrix S'' such that

(i) $((\phi_1 \cdot \sigma_{S'}) \cdot \sigma_{S''})(H) = H$;

(ii) $((\phi_1 \cdot \sigma_{S'}) \cdot \sigma_{S''})(\mathcal{L}_{st}) \subseteq \mathbf{n}^+$ for any $1 \leq s < t \leq n$.

Let

$$S_2 = S''S'.$$

Then by Lemma 2.1(2), the proofs of (1) and (2) are completed.

In the remainder of this proof, we denote

$$\phi_2 = \phi_1 \cdot \sigma_{S_2}.$$

Step 4. *For any $s \in \{1, 2, \cdots, n-1\}$, there is some $j \in \{1, 2, \cdots, n-1\}$ such that*

$$\phi_2(\mathcal{L}_{s,s+1}) = \mathcal{L}_{j,j+1}.$$

By Step 3,

$$\phi_2(\mathbf{n}^+) \subseteq \mathbf{n}^+.$$

Since $\dim\phi_2(\mathbf{n}^+) = \dim\mathbf{n}^+$, we have

$$\phi_2(\mathbf{n}^+) = \mathbf{n}^+.$$

Since $\mathbf{n}^+$ is a solvable subalgebra generated by all $\mathcal{L}_{s,s+1}$ for $s \in \{1, 2, \cdots, n-1\}$, we see that $\mathbf{n}^+$ is also generated by all $\phi_2(\mathcal{L}_{s,s+1})$ for $s \in \{1, 2, \cdots, n-1\}$. Then Step 4 holds from Step 2.

Step 5. *There is a graph automorphism ω of $gl(n,\mathbb{C})$ such that*

$$(\omega\cdot\phi_2)(\mathcal{L}_{s,s+1})=\mathcal{L}_{s,s+1}$$

for any $s\in\{1,2,\cdots,n-1\}$.

For any two distinct $s,t\in\{1,2,\cdots,n-1\}$, $|s-t|=1$ if and only if the dimension of the solvable subalgebra generated by $\mathcal{L}_{s,s+1}$ and $\mathcal{L}_{t,t+1}$ is 3, and $|s-t|>1$ if and only if the dimension of the solvable subalgebra generated by $\mathcal{L}_{s,s+1}$ and $\mathcal{L}_{t,t+1}$ is 2. By Step 4, we can set π to be the permutation of $\{1,2,\cdots,n-1\}$ such that

$$\phi_2(\mathcal{L}_{s,s+1})=\mathcal{L}_{\pi(s),\pi(s)+1}$$

for any $s=1,2,\cdots,n-1$. Since the dimension of the solvable subalgebra generated by $\mathcal{L}_{s,s+1}$ and $\mathcal{L}_{t,t+1}$ is equal to the dimension of the solvable subalgebra generated by $\mathcal{L}_{\pi(s),\pi(s)+1}$ and $\mathcal{L}_{\pi(t),\pi(t)+1}$, $|s-t|=1$ if and only if $|\pi(s)-\pi(t)|=1$, and $|s-t|>1$ if and only if $|\pi(s)-\pi(t)|>1$. Then either

(1) $\pi(s)=s$, $1\le s\le n-1$, or

(2) $\pi(s)=n-s$, $1\le s\le n-1$.

For the case (1), we set $\omega=I$; and for the case (2), we set $\omega=\omega_0$. Then Step 5 holds.

Denote

$$\phi_3=\omega\cdot\phi_2.$$

Step 6. $\phi_3(\mathcal{L}_{st})=\mathcal{L}_{st}$ *for any $s,t\in\{1,2,\cdots,n\}$ and $s\neq t$.*

At first we prove that

$$\phi_3(\mathcal{L}_{st})=\mathcal{L}_{st}\qquad\text{for any } 1\le s<t\le n.$$

To achieve the aim we use decreasing induction on $t-s$, where $1\le t-s\le n-1$. For $t-s=n-1$, then $t=n, s=1$. Since $\mathcal{L}_{1n}+\mathcal{L}_{k,k+1}$ is a two-dimensional solvable subalgebra for any $k=1,2,\cdots,n-1$, the image $\phi_3(\mathcal{L}_{1n}+\mathcal{L}_{k,k+1})$ is also a two-dimensional subalgebra, which is generated by $\phi_3(\mathcal{L}_{1n})$ and $\phi_3(\mathcal{L}_{k,k+1})$. Assume that

$$\phi_3(\mathcal{L}_{1n})=\mathcal{L}_{pq}\neq\mathcal{L}_{1n}.$$

Then there is some $i\in\{1,2,\cdots,n-1\}$ such that

$$[\mathcal{L}_{pq},\ \mathcal{L}_{i,i+1}]\neq 0,$$

i.e.,

$$[\phi_3(\mathcal{L}_{1n}),\ \phi_3(\mathcal{L}_{i,i+1})]\neq 0,$$

which implies that the subalgebra of $gl(n,\mathbb{C})$ generated by $\phi_3(\mathcal{L}_{1n})$ and $\phi_3(\mathcal{L}_{i,i+1})$ is at least three-dimensional, a contradiction. Thus

$$\phi_3(\mathcal{L}_{1n})=\mathcal{L}_{1n}.$$

Assume that

$$\phi_3(\mathcal{L}_{st})=\mathcal{L}_{st}$$

for any pair (s,t) satisfying $t-s\ge k+1$, $1\le s<t\le n$. Let (p,q) be a pair satisfying $|p-q|=k$ and $1\le p<q\le n$. There is some $i\in\{1,2,\cdots,n-1\}$ such that

$$[\mathcal{L}_{pq},\ \mathcal{L}_{i,i+1}]\neq 0.$$

The subalgebra $\mathfrak{t}$ generated by $\mathcal{L}_{pq}$ and $\mathcal{L}_{i,i+1}$ is

$$\mathcal{L}_{pq}+\mathcal{L}_{i,i+1}+[\mathcal{L}_{pq},\ \mathcal{L}_{i,i+1}],$$

which is three-dimensional and solvable. We consider the three-dimensional solvable algebra $\phi_3(\mathfrak{t})$. On the one hand, it is the subalgebra generated by $\phi_3(\mathcal{L}_{pq})$ and $\mathcal{L}_{i,i+1}$, i.e., it is the subalgebra

$$\mathcal{L}_{i,i+1} + \phi_3(\mathcal{L}_{pq}) + [\mathcal{L}_{i,i+1},\ \phi_3(\mathcal{L}_{pq})].$$

On the other hand, it is

$$\phi_3(\mathcal{L}_{i,i+1} + \mathcal{L}_{pq} + [\mathcal{L}_{i,i+1}, \mathcal{L}_{pq}]) = \mathcal{L}_{i,i+1} + \phi_3(\mathcal{L}_{pq}) + \phi_3([\mathcal{L}_{i,i+1}, \mathcal{L}_{pq}]).$$

By hypothesis,

$$\phi_3([\mathcal{L}_{i,i+1}, \mathcal{L}_{pq}]) = [\mathcal{L}_{i,i+1}, \mathcal{L}_{pq}].$$

Then

$$[\mathcal{L}_{i,i+1}, \mathcal{L}_{pq}] = [\mathcal{L}_{i,i+1}, \phi_3(\mathcal{L}_{pq})],$$

and so

$$\phi_3(\mathcal{L}_{pq}) = \mathcal{L}_{pq}.$$

By induction,

$$\phi_3(\mathcal{L}_{st}) = \mathcal{L}_{st} \qquad \text{for any } 1 \leq s < t \leq n.$$

It is easy to see that

$$\phi_3(\mathbf{n}^-) = \mathbf{n}^-, \qquad \phi_3(H) = H.$$

As in Step 4, we can similarly prove that for any $1 \leq i \leq n-1$, there is some j such that

$$\phi_3(\mathcal{L}_{i+1,i}) = \mathcal{L}_{j+1,j}.$$

For a given $i \in \{1, 2, \cdots, n-1\}$, if the above $j \neq i$, then the solvability of $\mathcal{L}_{j+1,j} + \mathcal{L}_{i,i+1} + H$ will force

$$\phi_3^{-1}(\mathcal{L}_{j+1,j} + \mathcal{L}_{i,i+1} + H) = \mathcal{L}_{i+1,i} + \mathcal{L}_{i,i+1} + H$$

to be solvable, absurd. So

$$\phi_3(\mathcal{L}_{i+1,i}) = \mathcal{L}_{i+1,i} \qquad \text{for any } 1 \leq i \leq n-1.$$

A similar discussion to the above shows that

$$\phi_3(\mathcal{L}_{st}) = \mathcal{L}_{st} \qquad \text{for any } 1 \leq t < s \leq n.$$

Step 7. *There exist a constant $v \in \mathbb{C}^*$ and a linear function f' such that*

$$(\psi_{f'} \cdot \psi_v^{-1} \cdot \phi_3)(h) = h \qquad \textit{for any } h \in H.$$

Let

$$\phi_3(E_{ii}) = \mathrm{diag}\{\lambda_{1i}, \lambda_{2i}, \cdots, \lambda_{ni}\} \qquad \text{for any } 1 \leq i \leq n,$$

and

$$\phi_3(E_{st}) = b_{st}E_{st} \qquad \text{for any } 1 \leq s \neq t \leq n.$$

For any fixed $i \in \{1, 2, \cdots, n\}$, and any two distinct $j, k \neq i$, E_{ii} and $E_{jk} + E_{kj}$ generate a two-dimensional solvable subalgebra of $gl(n, \mathbb{C})$. So $\phi_3(E_{ii})$ and $\phi_3(E_{kj} + E_{jk})$ also generate a two-dimensional subalgebra of $gl(n, \mathbb{C})$. Since

$$[\phi_3(E_{ii}),\ \phi_3(E_{kj} + E_{jk})] = b_{kj}(\lambda_{ki} - \lambda_{ji})E_{kj} + b_{jk}(\lambda_{ji} - \lambda_{ki})E_{jk},$$

we have

$$\lambda_{ki} = \lambda_{ji}$$

for any two distinct $j, k \neq i$. Thus

$$\lambda_{1i} = \lambda_{2i} = \cdots = \lambda_{i-1,i} = \lambda_{i+1,i} = \cdots = \lambda_{ni} \qquad \text{for any } i = 1, 2, \cdots, n.$$

So

$$\phi_3(E_{ii}) = (\lambda_{ii} - \lambda_{ki})E_{ii} + \lambda_{ki} I \qquad \text{for some } k \neq i.$$

We may assume that

$$\phi_3(E_{ii}) = v_i E_{ii} + u_i I \qquad \text{for any } i = 1, 2, \cdots, n,$$

where $u_i, v_i \in \mathbb{C}$. By Lemma 3.1,

$$\phi_3(I) = bI \qquad \text{for some } b \in \mathbb{C}^*.$$

Since

$$\phi_3(I) = \phi_3(E_{11}) + \phi_3(E_{22}) + \cdots + \phi_3(E_{nn}),$$

we have

$$v_1 = v_2 = \cdots = v_n,$$

denoted by v, and

$$b = v + \sum_{i=1}^{n} u_i.$$

Then

$$\phi_3(E_{ii}) = vE_{ii} + u_i I,$$

where $v \neq 0$ (otherwise, $\phi_3^{-1}(u_i I) = E_{ii}$, a contradiction to Lemma 3.1). So

$$(\psi_v^{-1} \cdot \phi_3)(E_{ii}) = E_{ii} + \frac{u_i}{v} I \qquad \text{for any } 1 \leq i \leq n.$$

Define a linear function

$$f' : gl(n, \mathbb{C}) \to \mathbb{C}$$

determined by

$$f'(E_{ii}) = -\frac{u_i}{b} \qquad \text{for any } 1 \leq i \leq n$$

and

$$f'(E_{st}) = 0 \qquad \text{for any } 1 \leq s \neq t \leq n.$$

So

$$f'(I) = -\frac{\sum\limits_{i=1}^{n} u_i}{b} = -1 + \frac{v}{b} \neq -1.$$

Correspondingly, there is an invertible linear map

$$\psi_{f'} : gl(n, \mathbb{C}) \to gl(n, \mathbb{C}), \qquad A \mapsto A + f'(A)I,$$

where $A \in gl(n, \mathbb{C})$. Then

$$(\psi_{f'} \cdot \psi_v^{-1} \cdot \phi_3)(E_{ii}) = E_{ii} \qquad \text{for any } 1 \leq i \leq n,$$

and so

$$(\psi_{f'} \cdot \psi_v^{-1} \cdot \phi_3)(h) = h \qquad \text{for any } h \in H,$$

and

$$(\psi_{f'} \cdot \psi_v^{-1} \cdot \phi_3)(\mathcal{L}_{st}) = \mathcal{L}_{st} \qquad \text{for any } 1 \leq s \neq t \leq n.$$

Denote

$$\phi_4 = \psi_{f'} \cdot \psi_v^{-1} \cdot \phi_3.$$

Step 8. *There is an invertible matrix S_3 such that*

$$(\sigma_{S_3} \cdot \phi_4)(A) = A \qquad \textit{for any } A \in gl(n, \mathbb{C}).$$

Suppose that

$$\phi_4(E_{i,i+1}) = b_i E_{i,i+1},$$

where $b_i \in \mathbb{C}^*$. Let

$$S_3 = \mathrm{diag}\{1, b_1^{-1}, b_1^{-1}b_2^{-1}, \cdots, b_1^{-1}b_2^{-1} \cdots b_{n-1}^{-1}\}.$$

Then

$$(\sigma_{S_3} \cdot \phi_4)(h) = h \qquad \text{for any } h \in H.$$

Let

$$(\sigma_{S_3} \cdot \phi_4)(E_{st}) = a_{st} E_{st} \qquad \text{for any } 1 \le s \ne t \le n,$$

where $a_{st} \in \mathbb{C}^*$.

At first we prove that

$$a_{st} = 1$$

for any pair (s, t) such that $1 \le s < t \le n$ by induction on $t - s$. It is easy to check that

$$(\sigma_{S_3} \cdot \phi_4)(E_{i,i+1}) = E_{i,i+1} \qquad \text{for any } 1 \le i \le n-1.$$

So it holds for $t - s = 1$. Assume that

$$a_{st} = 1 \qquad \text{for any } 1 < t - s < k.$$

We prove

$$a_{st} = 1$$

when $t - s = k$. Since

$$[E_{t-1,t-1} + E_{t-1,t}, E_{s,t-1} + E_{st}] = -(E_{s,t-1} + E_{st}),$$

$E_{t-1,t-1} + E_{t-1,t}$ and $E_{s,t-1} + E_{st}$ generate a two-dimensional solvable subalgebra, and so $(\sigma_{S_3} \cdot \phi_4)(E_{t-1,t-1} + E_{t-1,t})$ and $(\sigma_{S_3} \cdot \phi_4)(E_{s,t-1} + E_{st})$ also generate a two-dimensional solvable subalgebra, where

$$(\sigma_{S_3} \cdot \phi_4)(E_{t-1,t-1} + E_{t-1,t}) = E_{t-1,t-1} + E_{t-1,t},$$

and

$$(\sigma_{S_3} \cdot \phi_4)(E_{s,t-1} + E_{st}) = E_{s,t-1} + a_{st} E_{st}$$

by hypothesis. Since

$$[E_{t-1,t-1} + E_{t-1,t}, E_{s,t-1} + a_{st} E_{st}] = -(E_{s,t-1} + E_{st}),$$

we have

$$a_{st} = 1.$$

By induction,

$$a_{st} = 1 \qquad \text{for any } 1 \le s < t \le n.$$

Next, we prove that

$$a_{st} = 1 \qquad \text{for any } 1 \le t < s \le n.$$

First we prove that

$$a_{t+1,t} = 1 \qquad \text{for any } t = 1, 2, \cdots, n-1.$$

We prove it in the following two cases.

Case 1. $t = 1$.

Since

$$[E_{11} + E_{21}, E_{13} + E_{23}] = E_{13} + E_{23},$$

we see that $E_{11} + E_{21}$ and $E_{13} + E_{23}$ generate a two-dimensional solvable subalgebra of $gl(n, \mathbb{C})$, and so $(\sigma_{S_3} \cdot \phi_4)(E_{11} + E_{21})$ and $(\sigma_{S_3} \cdot \phi_4)(E_{13} + E_{23})$ also generate a two-dimensional subalgebra of $gl(n, \mathbb{C})$. Since

$$\begin{aligned} &[(\sigma_{S_3} \cdot \phi_4)(E_{11} + E_{21}),\ (\sigma_{S_3} \cdot \phi_4)(E_{13} + E_{23})] \\ =\ &[E_{11} + a_{21}E_{21},\ E_{13} + E_{23}] \\ =\ &E_{13} + a_{21}E_{23}, \end{aligned}$$

we have

$$a_{21} = 1.$$

Case 2. $t > 1$.

Since

$$[E_{t+1,t+1} + E_{t+1,t},\ E_{t-1,t+1} + E_{t-1,t}] = -(E_{t-1,t+1} + E_{t-1,t}),$$

we see that $E_{t+1,t+1} + E_{t+1,t}$ and $E_{t-1,t+1} + E_{t-1,t}$ generate a two-dimensional solvable subalgebra of $gl(n, \mathbb{C})$, and so $(\sigma_{S_3} \cdot \phi_4)(E_{t+1,t+1} + E_{t+1,t})$ and $(\sigma_{S_3} \cdot \phi_4)(E_{t-1,t+1} + E_{t-1,t})$ also generate a two-dimensional solvable subalgebra of $gl(n, \mathbb{C})$. Since

$$[(\sigma_{S_3} \cdot \phi_4)(E_{t+1,t+1} + E_{t+1,t}), (\sigma_{S_3} \cdot \phi_4)(E_{t-1,t+1} + E_{t-1,t})] = -(E_{t-1,t+1} + a_{t+1,t}E_{t-1,t}),$$

we have

$$a_{t+1,t} = 1.$$

A similar discussion as above shows that

$$a_{st} = 1 \qquad \text{for any } 1 \le t < s \le n.$$

Thus

$$(\sigma_{S_3} \cdot \phi_4)(E_{st}) = E_{st} \qquad \text{for any } 1 \le s, t \le n.$$

Thus Step 8 holds.

Step 9. *There are a nonzero element $\mu \in \mathbb{C}^*$, an invertible matrix S and a linear function f on $gl(n, \mathbb{C})$ with $f(I) \neq -\mu$ such that either*

$$\phi(X) = \mu S X S^{-1} + f(X)I$$

or

$$\phi(X) = \mu S X^t S^{-1} + f(X)I$$

for any $X \in gl(n, \mathbb{C})$.

By Step 8,

$$\mathbf{1} = \sigma_{S_3} \cdot \phi_4 = \sigma_{S_3} \cdot \psi_{f'} \cdot \psi_v^{-1} \cdot \omega \cdot \sigma_{S_1} \cdot \phi \cdot \sigma_{S_2},$$

and so

$$\phi = \sigma_{S_1}^{-1} \cdot \omega^{-1} \cdot \psi_v \cdot \psi_{f'}^{-1} \cdot \sigma_{S_3}^{-1} \cdot \sigma_{S_2}^{-1}.$$

We prove Step 9 in the following two cases:

Case 1. $\omega = 1$.

In this case, by Lemma 2.1, we have

$$\begin{aligned}\phi(X) &= (\sigma_{S_1^{-1}} \cdot \psi_v \cdot \psi_{f'}^{-1} \cdot \sigma_{S_2^{-1}S_3^{-1}})(X)\\ &= v(S_1S_3S_2)X(S_1S_3S_2)^{-1} - bf'(S_3S_2XS_2^{-1}S_3^{-1})I.\end{aligned}$$

Let

$$\mu = v, \qquad S = S_1S_3S_2,$$

and f be the linear function determined by

$$f(X) = -bf'(S_3S_2XS_2^{-1}S_3^{-1}).$$

Then

$$f(I) = -bf'(I) = b - v \neq -v = -u.$$

Thus Step 9 holds.

Case 2. $\omega = \omega_0$.

In this case, by Lemma 2.1, we have

$$\phi(X) = -v(S_1R(S_2^{-1}S_3^{-1})^t)X^t(S_1R(S_2^{-1}S_3^{-1})^t)^{-1} + bf'(S_3S_2XS_2^{-1}S_3^{-1})I.$$

Let

$$\mu = -v, \qquad S = S_1R(S_2^{-1}S_3^{-1})^t,$$

and f be the linear map determined by

$$f(X) = bf'(S_3S_2XS_2^{-1}S_3^{-1}).$$

Then

$$f(I) = bf'(I) = -b + v \neq v = -\mu.$$

Thus Step 9 holds.

Finally, Theorem 1.1 holds for the case $n \geq 3$.

Next we prove Theorem 1.1 holds for $n = 2$.

We only need to prove the essential direction.

Let ϕ be an invertible linear map preserving solvability on $gl(2, \mathbb{C})$.

Since

$$\mathcal{T}_2 = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} | a, b, c \in \mathbb{C} \right\}$$

is a solvable subalgebra of $gl(2, \mathbb{C})$, $\phi(\mathcal{T}_2)$ is a solvable subalgebra of $gl(2, \mathbb{C})$, and so there is an invertible matrix S_1 such that

$$\phi(\mathcal{T}_2) \subseteq S_1\mathcal{T}_2S_1^{-1},$$

which implies that

$$(\sigma_{S_1} \cdot \phi)(\mathcal{T}_2) = \mathcal{T}_2.$$

Here $\sigma_{S_1} \cdot \phi$ is still an invertible linear map preserving solvability on $gl(2, \mathbb{C})$. The set $gl(2, \mathbb{C})$ is a disjoint union of $\mathbb{C}I$, $\mathcal{N}$ and $\mathcal{D}'$, where $\mathcal{N}$ is the set of all matrices of the form $\lambda I + N$

with $N \neq 0$ and $N^2 = 0$, and $\mathcal{D}'$ is the set of all nonscalar diagonalizable matrices. By Section 3 of [11], each of the sets $\mathbb{C}I$, $\mathcal{N}$, and $\mathcal{D}'$ is invariant under $\sigma_{S_1} \cdot \phi$. Let

$$(\sigma_{S_1} \cdot \phi)(I) = \lambda_0 I, \qquad \lambda_0 \in \mathbb{C}^*.$$

Since

$$E_{12} \in \mathcal{N} \cap \mathcal{T}_2,$$

we have

$$(\sigma_{S_1} \cdot \phi)(E_{12}) \in \mathcal{N} \cap \mathcal{T}_2,$$

where

$$\mathcal{N} \cap \mathcal{T}_2 = \{\lambda I + \lambda' E_{12} | \lambda,\ \lambda' \in \mathbb{C}\}.$$

So we may assume that

$$(\sigma_{S_1} \cdot \phi)(E_{12}) = \lambda_1 I + t E_{12},$$

where $\lambda_1 \in \mathbb{C}$ and $t \in \mathbb{C}^*$. Similarly, by $E_{11} \in \mathcal{D}' \cap \mathcal{T}_2$, we have

$$(\sigma_{S_1} \cdot \phi)(E_{11}) \in \mathcal{D}' \cap \mathcal{T}_2,$$

and we may assume that

$$(\sigma_{S_1} \cdot \phi)(E_{11}) = aE_{11} + bE_{12} + cE_{22},$$

where $a, b, c \in \mathbb{C}$, and $a \neq c$. Choose

$$S_2 = I - \frac{b}{a-c} E_{12}.$$

It is easy to check that

$$(\sigma_{S_2} \cdot \sigma_{S_1} \cdot \phi)(E_{11}) = aE_{11} + cE_{22}, \qquad (\sigma_{S_2} \cdot \sigma_{S_1} \cdot \phi)(E_{12}) = \lambda_1 I + tE_{12}.$$

Denote

$$\phi_1 = \sigma_{S_2} \cdot \sigma_{S_1} \cdot \phi.$$

Then

$$\begin{aligned} \phi_1(E_{11}) &= (a-c)E_{11} + cI, \\ \phi_1(E_{22}) &= \phi_1(I) - \phi_1(E_{11}) = (a-c)E_{22} + (\lambda_0 - a)I, \end{aligned}$$

and so

$$\phi_1(H) = H.$$

Since E_{11}, E_{12} generate a two-dimensional solvable subalgebra, $\phi_1(E_{11})$ and $\phi_1(E_{12})$ also generate a two-dimensional solvable subalgebra. By computation,

$$[\phi_1(E_{11}),\ \phi_1(E_{12})] = [aE_{11} + cE_{22},\ \lambda_1 I + tE_{12}] = t(a-c)E_{12}.$$

Thus

$$\lambda_1 = 0,$$

which implies that

$$\phi_1(E_{12}) = tE_{12}.$$

Let

$$\phi_1(E_{21}) = h + a_{21}E_{21} + a_{12}E_{12},$$

where $h \in H$, $a_{12}, a_{21} \in \mathbb{C}$. Since E_{11}, E_{22}, E_{21} generate a three-dimensional solvable subalgebra, $\phi_1(E_{11})$, $\phi_1(E_{22})$ and $\phi_1(E_{21})$ generate a three-dimensional solvable subalgebra, denoted by $\mathfrak{t}$. Choose

$$h_1 = E_{11} + 2E_{22} \in H \subseteq \mathfrak{t}.$$

Then $(\text{ad } h_1)(\phi_1(E_{21})) \in \mathfrak{t}$, $(\text{ad } h_1)^2(\phi_1(E_{21})) \in \mathfrak{t}$, i.e., $a_{21}E_{21} - a_{12}E_{12} \in \mathfrak{t}$, $a_{21}E_{21} + a_{12}E_{12} \in \mathfrak{t}$. Thus $a_{21}E_{21} \in \mathfrak{t}$, $a_{12}E_{12} \in \mathfrak{t}$. If $a_{12} \neq 0$ (resp., $a_{21} \neq 0$), then $E_{12} \in \mathfrak{t}$ (resp., $E_{21} \in \mathfrak{t}$). Thus one of a_{12}, a_{21} is zero and the other is nonzero. Assume that

$$a_{12} \neq 0, \qquad a_{21} = 0,$$

i.e.,

$$\phi_1(E_{21}) = h + a_{12}E_{12}.$$

In this case $\phi_1(E_{12})$, $\phi_1(E_{21})$ and $\phi_1(E_{11} - E_{22})$ generate a solvable subalgebra of $gl(2, \mathbb{C})$, which contradicts the fact that the subalgebra generated by E_{21}, E_{12} and $E_{11} - E_{22}$ is not solvable. Thus

$$a_{21} \neq 0, \qquad a_{12} = 0,$$

i.e.,

$$\phi_1(E_{21}) = h + a_{21}E_{21}.$$

Next we prove that $h = 0$.

Assume that $h \neq 0$. Let

$$h = pE_{11} + qE_{22}, \qquad p, q \in \mathbb{C}.$$

We could choose $p', q' \in \mathbb{C}$ so that $p' \neq q'$, and $p'q \neq q'p$. Then $p'E_{11} + q'E_{22}$ and h are linearly independent. Let

$$h'' = \phi_1^{-1}(p'E_{11} + q'E_{22}) \in H.$$

Denote

$$\mathfrak{t}' = \phi_1(\mathbb{C}E_{21} + \mathbb{C}h'').$$

Since $\mathbb{C}E_{21} + \mathbb{C}h''$ is a two-dimensional solvable subalgebra generated by h'' and E_{21}, $\mathfrak{t}'$ is a two-dimensional solvable subalgebra generated by $\phi_1(E_{21})$ and $\phi_1(h'')$. However,

$$[\phi_1(E_{21}),\ \phi_1(h'')] = [h + a_{21}E_{21},\ p'E_{11} + q'E_{22}] = a_{21}(p' - q')E_{21},$$

a contradiction. Thus

$$h = 0,$$

i.e.,

$$\phi_1(E_{21}) = a_{21}E_{21},$$

where $a_{21} \in \mathbb{C}^*$. So

$$\begin{aligned}
(\psi_{a-c}^{-1} \cdot \phi_1)(E_{11}) &= E_{11} + \frac{c}{a-c}I, \\
(\psi_{a-c}^{-1} \cdot \phi_1)(E_{22}) &= E_{22} + \frac{\lambda_0 - a}{a-c}I, \\
(\psi_{a-c}^{-1} \cdot \phi_1)(E_{12}) &= \frac{t}{a-c}E_{12}, \\
(\psi_{a-c}^{-1} \cdot \phi_1)(E_{21}) &= \frac{a_{21}}{a-c}E_{21}.
\end{aligned}$$

Let f' be a linear function on $\mathbb{C}$ determined by

$$f'(E_{11}) = \frac{c}{a-c}, \qquad f'(E_{22}) = \frac{\lambda_0 - a}{a-c}, \qquad f'(E_{12}) = f'(E_{21}) = 0.$$

Since

$$f'(I) = \frac{\lambda_0}{a-c} - 1 \neq -1,$$

$$\psi_{f'} : gl(2,\mathbb{C}) \to gl(2,\mathbb{C}), \qquad A \mapsto A + f'(A)I,$$

is an invertible linear map preserving solvability. It is easy to check that

$$\begin{aligned}
(\psi_{f'}^{-1} \cdot \psi_{a-c}^{-1} \cdot \phi_1)(E_{11}) &= E_{11}, \\
(\psi_{f'}^{-1} \cdot \psi_{a-c}^{-1} \cdot \phi_1)(E_{22}) &= E_{22}, \\
(\psi_{f'}^{-1} \cdot \psi_{a-c}^{-1} \cdot \phi_1)(E_{12}) &= \frac{t}{a-c} E_{12}, \\
(\psi_{f'}^{-1} \cdot \psi_{a-c}^{-1} \cdot \phi_1)(E_{21}) &= \frac{a_{21}}{a-c} E_{21}.
\end{aligned}$$

Denote

$$\phi_2 = \psi_{f'}^{-1} \cdot \psi_{a-c}^{-1} \cdot \phi_1.$$

Choose

$$S_3 = E_{11} + \frac{a-c}{t} E_{22}.$$

Then

$$\begin{aligned}
(\sigma_{S_3} \cdot \phi_2)(E_{11}) &= E_{11}, \\
(\sigma_{S_3} \cdot \phi_2)(E_{22}) &= E_{22}, \\
(\sigma_{S_3} \cdot \phi_2)(E_{12}) &= E_{12}, \\
(\sigma_{S_3} \cdot \phi_2)(E_{21}) &= \frac{t a_{21}}{(a-c)^2} E_{21}.
\end{aligned}$$

Set

$$t' = \frac{t a_{21}}{(a-c)^2}.$$

We shall prove that

$$t' = 1.$$

Since

$$[E_{11} + E_{12} - E_{21} - E_{22},\ E_{21} + E_{12}] = 2(E_{11} + E_{12} - E_{21} - E_{22}),$$

we see that $E_{11} + E_{12} - E_{21} - E_{22}$ and $E_{21} + E_{12}$ generate a two-dimensional solvable subalgebra of $gl(2,\mathbb{C})$, and so $(\sigma_{S_3} \cdot \phi_2)(E_{11} + E_{12} - E_{21} - E_{22})$ and $(\sigma_{S_3} \cdot \phi_2)(E_{21} + E_{12})$ also generate a two-dimensional solvable subalgebra of $gl(2,\mathbb{C})$. By computation,

$$\begin{aligned}
&[(\sigma_{S_3} \cdot \phi_2)(E_{11} + E_{12} - E_{21} - E_{22}),\ (\sigma_{S_3} \cdot \phi_2)(E_{21} + E_{12})] \\
&= 2(t'E_{11} + E_{12} - t'E_{21} - t'E_{22}).
\end{aligned}$$

Thus $t' = 1$, i.e.,

$$(\sigma_{S_3} \cdot \phi_2)(E_{21}) = E_{21}.$$

Therefore

$$\sigma_{S_3} \cdot \phi_2 = \mathbf{1},$$

i.e.,

$$\sigma_{S_3} \cdot \psi_{f'}^{-1} \cdot \psi_{a-c}^{-1} \cdot \sigma_{S_2} \cdot \sigma_{S_1} \cdot \phi = \mathbf{1}.$$

Thus by Lemma 2.1,

$$\phi = \sigma_{S_1}^{-1} \cdot \sigma_{S_2}^{-1} \cdot \psi_{a-c} \cdot \psi_{f'} \cdot \sigma_{S_3}^{-1} = \sigma_{S_2^{-1}S_1^{-1}} \cdot \psi_{a-c} \cdot \psi_{f'} \cdot \sigma_{S_3^{-1}}.$$

Therefore

$$\phi(X) = (a-c)(S_1S_2S_3)X(S_1S_2S_3)^{-1} + (a-c)f'(S_3XS_3^{-1})I$$

for any $X \in gl(2, \mathbb{C})$. Let

$$\mu = a - c, \qquad S = S_1S_2S_3,$$

and f be the linear function determined by

$$f(X) = (a-c)f'(S_3XS_3^{-1}).$$

So

$$f(I) = (a-c)f'(I) \neq -(a-c) = -\mu.$$

Thus

$$\phi(X) = \mu SXS^{-1} + f(X)I,$$

and Theorem 1.1 holds for $n = 2$.

The proof of Theorem 1.1 is completed.

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