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Superconvergence Points of Several Polynomial and Nonpolynomial Hermite Spectral Interpolations

Min Wang[1] and Zhimin Zhang[2,*]

[1] *China University of Mining and Technology, Beijing 10083, P.R. China.*
[2] *Department of Mathematics, Wayne State University, Detroit, Michigan 48202, USA.*

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Abstract. In this paper, we analyze the superconvergence properties for spectral interpolations by Hermite polynomials and mapped Hermite functions. At the superconvergence points, the $(N-k)$-th term in the Hermite spectral interpolation remainder for the $(k+1)$-th derivatives vanish. To solve multi-point weakly singular nonlocal problems, we previously introduced mapped Hermite functions (MHFs), which are constructed by applying a mapping to the Hermite polynomials. We prove that the superconvergence points of the spectral interpolations based on MHFs for the $(k+1)$-th derivatives are the zero points of the $(N-k)$-th term. Additionally, due to the rapid growth of the logarithmic function at the endpoints 0 and 1, we further propose generalized mapped Hermite functions (GMHFs). We develop basic approximation theory for these new orthogonal functions and prove the projection error and interpolation error in the L^2-weighted space using the pseudo-derivative. We demonstrate that the superconvergence points of the spectral interpolations based on both MHFs and GMHFs for the $(k+1)$-th derivative are the zero points of the $(N-k)$-th term. Numerical experiments confirm our theoretical results.

AMS subject classifications: 65N35, 65M70, 41A05

Key words: Superconvergence points, interpolation, Hermite polynomials, mapped Hermite functions, generalized mapped Hermite functions.

1 Introduction

The study of superconvergence phenomenon for h-version methods has had a significant impact on scientific computing, particularly on a posteriori error estimates and adaptive methods. As for the p-version methods and spectral methods, the initial studies, presented in [34, 42, 43], discussed some special and simple cases. The superconvergence

*Corresponding author. *Email addresses:* `minwang@cumtb.edu.cn` (M. Wang), `ag7761@wayne.edu` (Z. Zhang)

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properties of some high-order orthogonal polynomial interpolations were studied, and the superconvergence points were identified in [44]. These results were subsequently extended to general Jacobi-Gauss-type interpolation in [35]. Zhao and Zhang [45] investigated the superconvergence points of spectral interpolation involving fractional derivatives. The superconvergence properties of the Riemann-Liouville fractional derivative of Hermite interpolations were explored in [10]. Xiang *et al.* [38] rigorously showed that the Jacobi expansion for a more general class of ϕ-functions also exhibits a local supconvergence behaviour.

The Hermite spectral interpolation, especially in the spectral Galerkin method [18] and numerical quadratures, is widely applied in mathematical models involving nonlocal operators such as fractional integrals, fractional Laplacians, and nonlocal Laplacians. These methods have proven to be of great value and superior to conventional models in simulating many abnormal physical phenomena and engineering processes [9, 11]. We remark that there has been much interest in Hermite spectral methods for PDEs involving standard or integral fractional Laplacian in unbounded domains [21, 30]. In these methods using Hermite polynomial/function-based approaches, selecting more accurate points is a critical concern. One of the efforts in this work is dedicated to identifying superconvergence points for the $(k+1)$-th derivatives of the interpolants using Hermite polynomials/functions.

Nonlocal models, such as weakly singular integral equations (WSIEs), have been demonstrated to be more effective in modeling complex systems in physics, finance, and other fields [1, 17, 23, 24]. Due to the nonlocal nature of the weakly singular integral operator, existing methods such as collocation methods, Petrov-Galerkin methods, discontinuous Galerkin methods, and fast algorithms [2–5, 12, 20, 29, 31, 33, 36, 39, 40] have been applied. Spectral methods are promising candidates for solving WSIEs, as their global nature aligns well with the nonlocal definition of integral operators. Using integer-order orthogonal polynomials as basis functions, spectral methods [8, 16, 19, 25, 27] help reduce the memory cost associated with the discretization of fractional derivatives. Furthermore, to address singularities, the authors in [6, 7, 15, 41] have developed suitable basis functions.

One major challenge is to construct an effective basis for a spectral scheme to handle singularities. With a suitable basis, one can then analyze the approximation error to identify superconvergence points. The mapped Hermite spectral interpolation [37], which is based on mapped Hermite functions constructed by applying a mapping to Hermite polynomials, is specifically designed to match the multiple singularities present in the underlying solutions of weakly singular problems. In [37], we proposed a MHFs-spectral collocation method and a MHFs-smoothing transformation method to solve the two-point weakly singular Fredholm-Hammerstein integral equation directly, rather than indirectly by splitting the Fredholm integral kernel into two Volterra integral kernels. However, the mapping $\alpha\log(x/(1-x))$ used in the mapped Hermite functions grows rapidly near the endpoints $x=0$ and $x=1$, which may affect the accuracy in some situations. To address this issue, in this work we propose the generalized mapped Hermite functions by multiplying MHFs by $x^{\beta/2}(1-x)^{\beta/2}$. The study of the superconvergence properties for these nonpolynomial spectral interpolations is lacking. In this work, we also focus on considering the derivatives of nonpolynomial interpolants and identifying those points where function values are superconvergent. In this part, we study the superconvergence properties of both MHFs and GMHFs. When evaluating the results of problems, such as two-point weakly singular partial differential equations and integral equations using the corresponding nonpolynomial interpolants, the method here suggests better points where higher accuracy is achieved.

This paper is organized as follows. In Section 2, we review the Hermite polynomials/functions and identify superconvergence points for interpolating analytic functions. We introduce the MHFs Section 3 and locate superconvergence points of the interpolants for the derivatives. In Section 4, we propose the GMHFs and derive optimal projection and interpolation errors in weighted pseudo-derivatives that are adapted to the involved mapping. We compare the projection errors between MHFs and GMHFs and identify the superconvergence points of the interpolants by GMHFs for the derivatives and pseudo-derivatives. The last section is dedicate to numerical experiments and concluding remarks.

2 Hermite polynomials

In this section, we are concerned with the superconvergence of derivatives of Hermite spectral interpolants under an analytic assumption. First, we review some basic facts of Hermite polynomials. For more information about Hermite polynomials, we refer to [22, 26, 28]. Let $\omega(x)=e^{-x^2}$ be the Hermite weight function and let $\mathbb{N}_0:=\{0,1,2,\ldots\}$. For each $n\in\mathbb{N}_0$, the Hermite polynomial of degree n is defined by

$$H_n(x)=\frac{(-1)^n}{\omega(x)}\frac{\mathrm{d}^n\omega(x)}{\mathrm{d}x^n},$$

and is orthogonal with respect to $\omega(x)$ on $(-\infty,+\infty)$, namely,

$$\int_{-\infty}^{+\infty}H_m(x)H_n(x)\omega(x)dx=\gamma_n\delta_{mn},\quad \gamma_n=\sqrt{\pi}2^n n!,$$

where δ_{mn} is the Kronecker delta. The Hermite polynomials satisfy

$$H_n'(x)=2nH_{n-1}(x),\quad n\geqslant 1. \tag{2.1}$$

The Hermite contour integral will be introduced. First, an infinite strip which characterizes the domain of convergence of a Hermite series in the complex plane is introduced

$$\mathcal{S}_\rho:=\{z\in\mathbb{C}:\Im(z)\in[-\rho,\rho]\}, \tag{2.2}$$

where $\rho>0$ is determined by the location of singularities and the asymptotic behavior at infinity of the underlying function [13, 14].

The Hermite spectral interpolation is defined as

$$I_N^h f(x_j)=f(x_j), \tag{2.3}$$

where $I_N^h f(x)$ is the unique polynomial of degree N which interpolates $f(x)$ at the zero points $\{x_k\}_{k=0}^N$ of Hermite polynomials $H_{N+1}(x)$. Then the contour integral representation of the remainder for the above interpolation is introduced.

Lemma 2.1 ([32]). *If f is analytic in the infinite strip $\mathcal{S}_\rho$ for some $\rho>0$ and $|f(x)|\le\mathcal{K}|x|^\sigma$ for some $\sigma\in\mathbb{R}$ as $|x|\to\infty$ within the strip. For $x\in\mathbb{R}$ and $N\ge\max\{\lfloor\sigma\rfloor,0\}$, we have*

$$R_N^h(x):=f(x)-I_N^h f(x)=\frac{1}{2\pi\mathrm{i}}\int_{\partial\mathcal{S}_\rho}\frac{H_{N+1}(x)f(z)}{H_{N+1}(z)(z-x)}\mathrm{d}z, \tag{2.4}$$

where $I_N^h f$ is the polynomial of degree N determined through (2.3).

By direct differentiation of (2.4) and in view of (2.1), we can obtain the error equation for the derivative

$$\begin{aligned}\big(f(x)-I_N^h f(x)\big)'&=\left(\frac{1}{2\pi\mathrm{i}}\int_{\partial\mathcal{S}_\rho}\frac{H_{N+1}(x)f(z)}{H_{N+1}(z)(z-x)}\mathrm{d}z\right)'\\&=\frac{1}{2\pi i}\int_{\partial\mathcal{S}_\rho}\left(\frac{(2N+2)H_N(x)}{(z-x)}+\frac{H_{N+1}(x)}{(z-x)^2}\right)\frac{f(z)}{H_{N+1}(z)}\mathrm{d}z,\end{aligned} \tag{2.5}$$

where $I_N^h f(x)$ is defined in (2.3), and the second derivative

$$\begin{aligned}&\big(f(x)-I_N^h f(x)\big)''\\=&\frac{1}{2\pi i}\int_{\partial\mathcal{S}_\rho}\left(\frac{(2N+2)2NH_{N-1}(x)}{(z-x)}+\frac{2(2N+2)H_N(x)}{(z-x)^2}+\frac{2H_{N+1}(x)}{(z-x)^3}\right)\frac{f(z)}{H_{N+1}(z)}\mathrm{d}z.\end{aligned} \tag{2.6}$$

We can find that the magnitude of the first term in (2.5) is larger than that of the other term by a factor of about N. And in (2.6), the magnitude of the first term is larger than that of other terms. Figs. 1 and 2 give an intuitive representation. Namely, the zero points of $H_{N-k}(x)$, $k=0,1$, can be considered as the superconvergence points of the $(k+1)$-th derivatives of Hermite spectral interpolations. Taking the zero points $\{\xi_h^i\}_{i=0}^{N-1}$ of $H_N(x)$ as the superconvergence points, the term $(2N+2)H_N(x)$ vanishes. Thus, we have

$$\big(f(\xi_h^i)-I_N^h f(\xi_h^i)\big)'=\frac{1}{2\pi i}\int_{\partial\mathcal{S}_\rho}\frac{H_{N+1}(\xi_h^i)}{\left(z-\xi_h^i\right)^2}\frac{f(z)}{H_{N+1}(z)}\mathrm{d}z.$$

Take the zero points $\{\xi_h^i\}_{i=0}^{N-2}$ of $H_{N-1}(x)$ as the superconvergence points and the term $(2N+2)2NH_{N-1}(x)$ vanishes. Then we have

$$\big(f(\xi_h^i)-I_N^h f(\xi_h^i)\big)''=\frac{1}{2\pi i}\int_{\partial\mathcal{S}_\rho}\left(\frac{2(2N+2)H_N\left(\xi_h^i\right)}{\left(z-\xi_h^i\right)^2}+\frac{2H_{N+1}\left(\xi_h^i\right)}{\left(z-\xi_h^i\right)^3}\right)\frac{f(z)}{H_{N+1}(z)}\mathrm{d}z.$$

Furthermore, we can obtain the corresponding results for all $k\ll N$.

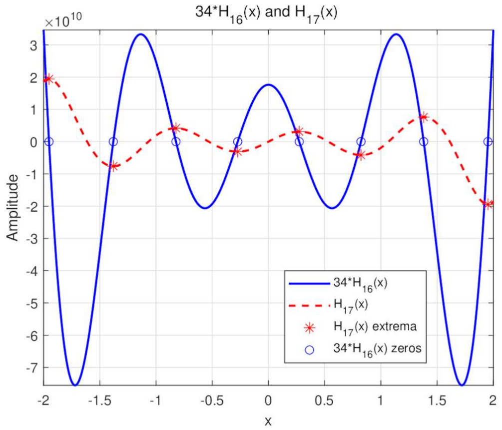

Figure 1: The magnitude of $34H_{16}(x)$ and $H_{17}(x)$ of $N=16$.

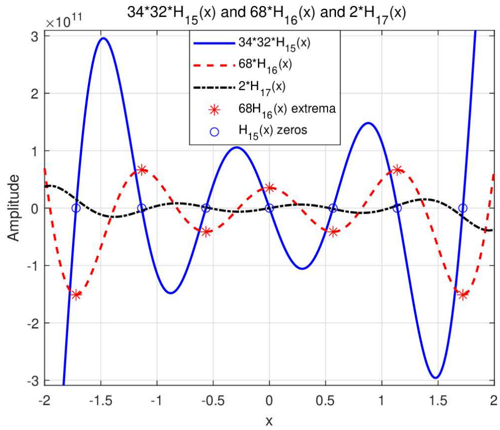

Figure 2: The magnitude of $34\times 32H_{15}(x)$ and $68H_{16}(x)$ and $2H_{17}(x)$ of $N=16$.

Theorem 2.1. *Let $\mathcal{S}_\rho$ be the infinite strip defined in* (2.2)*. If f is analytic in the infinite strip $\mathcal{S}_\rho$ for some $\rho>0$ and $|f(z)|\le\mathcal{K}|z|^\sigma$ for some $\sigma\in\mathbb{R}$ as $|z|\to\infty$ within the strip, then consider $I_N^h f(x)$ to be the Hermite spectral interpolant of degree N defined in* (2.3)*. For $k\ll N$, the superconvergence points of $\partial_x^{k+1}(f(x)-I_N^h f(x))$ are the zero points of $H_{N-k}(x)$.*

Proof.

$$
\begin{aligned}
&\partial_x^{k+1}\left(f(x)-I_N^h f(x)\right)\\
=&\frac{1}{2\pi i}\int_{\partial\mathcal{S}_\rho}\partial_x^{k+1}\left(\frac{H_{N+1}(x)}{z-x}\right)\frac{u(z)}{H_{N+1}(z)}\mathrm{d}z\\
=&\frac{1}{2\pi i}\sum_{l=0}^{k+1}\binom{k+1}{l}\int_{\partial\mathcal{S}_\rho}\frac{\partial_x^l H_{N+1}(x)}{(z-x)^{k-l+2}}\frac{u(z)}{H_{N+1}(z)}\mathrm{d}z\\
=&\frac{1}{2\pi i}\sum_{l=0}^{k+1}\binom{k+1}{l}\int_{\partial\mathcal{S}_\rho}\frac{2(N+1)2N\cdots 2(N+2-l)H_{N+1-l}(x)}{(z-x)^{k-l+2}}\frac{u(z)}{H_{N+1}(z)}\mathrm{d}z.
\end{aligned}\tag{2.7}
$$

The magnitude of the term $2(N+1)2N\cdots 2(N+1-k)H_{N-k}(x)$ with $l=k+1$ is larger than that of the other terms $2(N+1)2N\cdots 2(N-l)H_{N+1-l}(x), l=0,\ldots,k$. Let $\{\xi_h^i\}_{i=0}^{N-k-1}$ be the zero points of $H_{N-k}(x)$ and we have

$$
\begin{aligned}
&\partial_x^{k+1}\left(f(\xi_h^i)-I_N^h f(\xi_h^i)\right)\\
=&\frac{1}{2\pi i}\sum_{l=0}^{k}\binom{k+1}{l}\int_{\partial\mathcal{S}_\rho}\frac{2(N+1)2N\cdots 2(N+2-l)H_{N+1-l}(\xi_h^i)}{\left(z-\xi_h^i\right)^{k-l+2}}\frac{u(z)}{H_{N+1}(z)}\mathrm{d}z.
\end{aligned}\tag{2.8}
$$

At these points, $2(N+1)2N\cdots 2(N+1-k)H_{N-k}(x)$ disappears. Thus, these points are the superconvergence points of $\partial_x^{k+1}(f(x)-I_N^h f(x))$. □

The Hermite function is defined by

$$\psi_n = \frac{e^{-x^2/2} H_n(x)}{\sqrt{\gamma_n}}, \quad n \in \mathbb{N}_0. \tag{2.9}$$

The Hermite functions satisfy

$$\psi_n'(x) = \sqrt{2n}\psi_{n-1}(x) - x\psi_n(x). \tag{2.10}$$

We consider the Hermite spectral interpolation using Hermite functions

$$\hat{I}_N^h f(x_j) = f(x_j), \tag{2.11}$$

where $\hat{I}_N^h f(x) \in \mathbb{H}_N := \mathrm{span}\{\psi_k\}_{k=0}^N$ and $\{x_j\}_{j=0}^N$ are the zeros of $\psi_{N+1}(x)$. Considering (2.11), we observe that $e^{x^2/2}\hat{I}_N^h f(x)$ is a polynomial of degree N which interpolates $e^{x^2/2} f(x)$ at the points $\{x_j\}_{j=0}^N$ which are the zero points of $H_{N+1}(x)$. Combining this observation with (2.4) yields

$$f(x) - \hat{I}_N^h f(x) = \frac{\sqrt{\gamma_{N+1}}}{2\pi \mathrm{i}} \int_{\partial \mathcal{S}_\rho} \frac{\psi_{N+1}(x) e^{z^2/2} f(z)}{H_{N+1}(z)(z-x)} \mathrm{d}z. \tag{2.12}$$

In view of (2.10) and (2.12), we have

$$\begin{aligned} \left(f(x) - \hat{I}_N^h f(x)\right)' &= \frac{\sqrt{\gamma_{N+1}}}{2\pi i} \left(\int_{\partial \mathcal{S}_\rho} \frac{\psi_{N+1}(x) e^{z^2/2} f(z)}{H_{N+1}(z)(z-x)^2} \mathrm{d}z + \int_{\partial \mathcal{S}_\rho} \frac{\psi_{N+1}'(x) e^{z^2/2} f(z)}{H_{N+1}(z)(z-x)} \mathrm{d}z \right) \\ &= \frac{\sqrt{\gamma_{N+1}}}{2\pi i} \left(\int_{\partial \mathcal{S}_\rho} \frac{\psi_{N+1}(x) e^{z^2/2} f(z)}{(z-x)^2 H_{N+1}(z)} + \frac{\sqrt{2(N+1)}\psi_N(x) e^{z^2/2} f(z)}{(z-x) H_{N+1}(z)} \right. \\ &\qquad \left. - \frac{x\psi_{N+1}(x) e^{z^2/2} f(z)}{(z-x) H_{N+1}(z)} \mathrm{d}z \right), \end{aligned} \tag{2.13}$$

$$\begin{aligned} \left(f(x) - \hat{I}_N^h f(x)\right)'' = \frac{\sqrt{\gamma_{N+1}}}{2\pi i} \int_{\partial \mathcal{S}_\rho} & \left(\frac{2\psi_{N+1}(x)}{(z-x)^3} + \frac{2\sqrt{2(N+1)}\psi_N(x)}{(z-x)^2} - \frac{2x\psi_{N+1}(x)}{(z-x)^2} \right. \\ & + \frac{\sqrt{4(N+1)N}\psi_{N-1}(x)}{z-x} - \frac{2x\sqrt{2(N+1)}\psi_N(x)}{z-x} \\ & \left. - \frac{\psi_{N+1}(x)}{z-x} + \frac{x^2\psi_{N+1}(x)}{z-x} \right) \frac{e^{z^2/2} f(z)}{H_{N+1}(z)} \mathrm{d}z. \end{aligned} \tag{2.14}$$

We can find that in (2.13), the magnitude of the second term with $\sqrt{2(N+1)}\psi_N(x)$ is larger than that of other terms. Similarly, the magnitude of the fourth term in (2.14) is the largest. Figs. 3 and 4 provide a visual representation. The function value approximations are optimal at the roots of $\psi_N(x)$ and $\psi_{N-1}(x)$, respectively. Specifically, by taking the

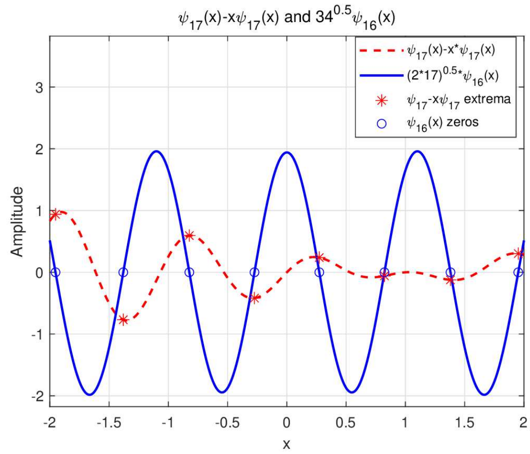

Figure 3: The magnitude of $\psi_{17}(x)-x\psi_{17}(x)$ and $\sqrt{34}\psi_{16}(x)$ and $x\psi_{17}(x)$ of $N=16$.

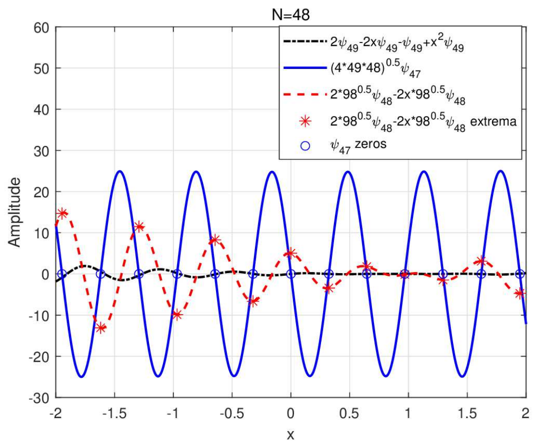

Figure 4: The magnitude of $2\psi_{49}-2x\psi_{49}-\psi_{49}+x^2\psi_{49}$ and $(4\times 49\times 48)^{0.5}\psi_{47}$ and $2\times 98^{0.5}\psi_{48}-2x\times 98^{0.5}\psi_{48}$ of $N=48$.

zero points $\{\xi_h^i\}_{i=0}^{N-1}$ of $\psi_N(x)$ as the superconvergence points, the term $\sqrt{2(N+1)}\psi_N(x)$ becomes zero,

$$\begin{aligned}&\left(f(\xi_h^i)-\hat{I}_N^h f(\xi_h^i)\right)'\\ =&\frac{\sqrt{\gamma_{N+1}}}{2\pi i}\left(\int_{\partial\mathcal{S}_\rho}\frac{\psi_{N+1}(\xi_h^i)e^{z^2/2}f(z)}{H_{N+1}(z)\left(z-\xi_h^i\right)^2}\mathrm{d}z-\int_{\partial\mathcal{S}_\rho}\frac{\xi_h^i\psi_{N+1}(\xi_h^i)e^{z^2/2}f(z)}{H_{N+1}(z)\left(z-\xi_h^i\right)}\mathrm{d}z\right).\end{aligned}$$

Take the zero points $\{\xi_h^i\}_{i=0}^{N-2}$ of $\psi_{N-1}(x)$ as the superconvergence points, and the term $\sqrt{4(N+1)N}\psi_{N-1}(x)$ vanishes

$$\begin{aligned}&\left(f(\xi_h^i)-\hat{I}_N^h f(\xi_h^i)\right)''\\ =&\frac{\sqrt{\gamma_{N+1}}}{2\pi i}\int_{\partial\mathcal{S}_\rho}\left(\frac{2\psi_{N+1}(\xi_h^i)}{\left(z-\xi_h^i\right)^3}+\frac{2\sqrt{2(N+1)}\psi_N(\xi_h^i)-2\xi_h^i\psi_{N+1}(\xi_h^i)}{\left(z-\xi_h^i\right)^2}\right.\\ &\left.+\frac{-2\xi_h^i\sqrt{2(N+1)}\psi_N(\xi_h^i)-\psi_{N+1}(\xi_h^i)+\left(\xi_h^i\right)^2\psi_{N+1}(\xi_h^i)}{z-\xi_h^i}\right)\frac{e^{z^2/2}f(z)}{H_{N+1}(z)}\mathrm{d}z.\end{aligned}$$

Further we derive the result for arbitrary $k\ll N$.

Theorem 2.2. *If f is analytic in the infinite strip $\mathcal{S}_\rho$ for some $\rho>0$ and $|f(z)|\le\mathcal{K}|z|^\sigma$ for some $\sigma\in\mathbb{R}$ as $|z|\to\infty$ within the strip, then consider $\hat{I}_N^h f(x)$ to be the Hermite spectral interpolation with Hermite functions defined in* (2.11). *For $k\ll N$, the superconvergence points of $\partial_x^{k+1}(f(x)-\hat{I}_N^h f(x))$ are the zero points of $\psi_{N-k}(x)$.*

Proof.

$$
\begin{aligned}
&\left|\left(f(x)-\hat{I}_N^h f(x)\right)^{(k+1)}\right| \\
=&\frac{\sqrt{\gamma_{N+1}}}{2\pi i}\int_{\partial\mathcal{S}_\rho}\partial_x^{k+1}\left(\frac{\psi_{N+1}(x)}{z-x}\right)\frac{e^{z^2/2}f(z)}{H_{N+1}(z)}\mathrm{d}z \\
=&\frac{\sqrt{\gamma_{N+1}}}{2\pi i}\sum_{l=0}^{k+1}\binom{k+1}{l}\int_{\partial\mathcal{S}_\rho}\frac{\partial_x^l\psi_{N+1}(x)}{(z-x)^{k-l+2}}\frac{e^{z^2/2}f(z)}{H_{N+1}(z)}\mathrm{d}z.
\end{aligned} \tag{2.15}
$$

In view of

$$
\begin{aligned}
\partial_x^l\psi_{N+1}(x)=&\sqrt{2(N+1)2N\cdots2(N+2-l)}\,\psi_{N+1-l}(x) \\
&+P_1(x)\sqrt{2(N+1)\cdots2(N+3-l)}\,\psi_{N-l+2}(x)+\cdots+P_l(x)\psi_{N+1}(x),
\end{aligned}
$$

where P_i, $i=1,\ldots,l+1$ are polynomials of degree i, we can obtain

$$
\begin{aligned}
&\left|\left(f(x)-\hat{I}_N^h f(x)\right)^{(k+1)}\right| \\
=&\frac{\sqrt{\gamma_{N+1}}}{2\pi i}\sum_{l=0}^{k+1}\binom{k+1}{l}\int_{\partial\mathcal{S}_\rho}\Bigg(\sqrt{2(N+1)2N\cdots2(N+2-l)}\,\psi_{N+1-l}(x)+\cdots \\
&+P_l(x)\psi_{N+1}(x)\Bigg)\frac{e^{z^2/2}f(z)}{(z-x)^{k-l+2}H_{N+1}(z)}\mathrm{d}z.
\end{aligned} \tag{2.16}
$$

The largest magnitude is that of the term $\sqrt{2(N+1)2N\cdots2(N+1-k)}\,\psi_{N-k}(x)$ in (2.16) with $l=k+1$. Take the zero points $\{\xi_h^i\}_{i=0}^{N-k-1}$ of $\psi_{N-k}(x)$ as the superconvergence points, and we have

$$
\begin{aligned}
&\left|\left(f(\xi_h^i)-\hat{I}_N^h f(\xi_h^i)\right)^{(k+1)}\right| \\
=&\frac{\sqrt{\gamma_{N+1}}}{2\pi i}\sum_{l=0}^{k}\binom{k+1}{l}\int_{\partial\mathcal{S}_\rho}\Bigg(\sqrt{2(N+1)2N\cdots2(N+2-l)}\,\psi_{N+1-l}(\xi_h^i)+\cdots \\
&+P_l(\xi_h^i)\psi_{N+1}(\xi_h^i)\Bigg)\frac{e^{z^2/2}f(z)}{(z-\xi_h^i)^{k-l+2}H_{N+1}(z)}\mathrm{d}z,
\end{aligned}
$$

in which the term with $l=k+1$ disappears. ☐

3 Mapped Hermite functions

In this section, we consider the superconvergence for derivatives of the mapped Hermite interpolants. First, we review the definition and the properties of mapped Hermite functions [37]

$$
\mathcal{Q}_n^{(\alpha)}(x)=H_n(z(x))=H_n\left(\alpha\log\left(\frac{x}{1-x}\right)\right),\quad n=0,1,\ldots, \tag{3.1}
$$

where H_n are the Hermite polynomials. Then we review the mapped Hermite functions interpolation

$$\mathcal{I}_N^{\alpha}: C(I) \;\rightarrow\; \mathcal{P}_N^{\log},$$

satisfies

$$\mathcal{I}_N^{\alpha} f\big(x_i^{(\alpha)}\big) = f\big(x_j^{(\alpha)}\big), \tag{3.2}$$

where $\mathcal{I}_N^{\alpha} f$ is the unique polynomial of degree N which interpolates $f(x)$ at the zero points $\{x_k\}_{k=0}^{N}$ of MHFs $\mathcal{Q}_{N+1}^{(\alpha)}(x)$.

Similarly to (2.2), we define the following domain:

$$\mathcal{T}_\rho := \left\{ \alpha \log\left(\frac{x}{1-x}\right) \in \mathbb{C} : \Im\left(\alpha \log\left(\frac{x}{1-x}\right)\right) \in [-\rho,\rho] \right\}, \tag{3.3}$$

and denote by $\partial\mathcal{T}_\rho$ the boundary of $\mathcal{T}_\rho$, see Fig. 5 for detail.

As shown in (3.1), the MHFs can be regarded as the transformation of Hermite polynomials. To obtain the error equation for the first and second derivatives, we take the following variable transformation:

$$x = \frac{e^{z/\alpha}}{1+e^{z/\alpha}}, \quad z \in (-\infty,\infty),$$

and denote

$$g(z) = f\left(\frac{e^{z/\alpha}}{1+e^{z/\alpha}}\right).$$

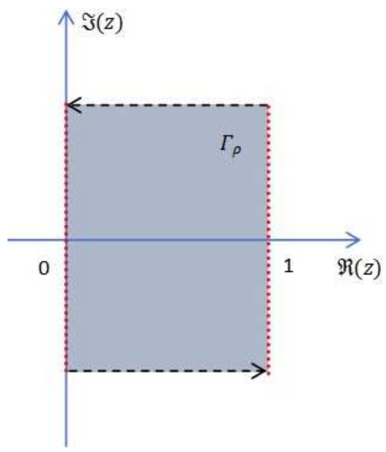

Figure 5: The strip $\partial\mathcal{T}_\rho$ and its boundary.

In view of (3.1) and (2.4), we have

$$
\begin{aligned}
&\partial_x\big(f(x)-\mathcal{I}_N^\alpha f(x)\big)\\
=&\partial_x\big(g(z)-I_N^h g(z)\big)=\frac{1}{2\pi i}\int_{\partial\mathcal{S}_\rho}\partial_x\left(\frac{H_{N+1}(z)}{y-z}\right)\left(\frac{v(y)}{H_{N+1}(y)}\right)\mathrm{d}y\\
=&\frac{1}{2\pi i}\int_{\partial\mathcal{S}_\rho}\left(\frac{H_{N+1}'(z)z'(x)}{(y-z)}+\frac{H_{N+1}(z)z'(x)}{(y-z)^2}\right)\left(\frac{v(y)}{H_{N+1}(y)}\right)\mathrm{d}y\\
=&\frac{1}{2\pi i}\int_{\partial\mathcal{S}_\rho}\left(\frac{2\alpha(N+1)\mathcal{Q}_N^{(\alpha)}(x)}{(y-z)x(1-x)}+\frac{\alpha\mathcal{Q}_{N+1}^{(\alpha)}(x)}{(y-z)^2x(1-x)}\right)\left(\frac{v(y)}{H_{N+1}(y)}\right)\mathrm{d}y.
\end{aligned}
\tag{3.4}
$$

In (3.4), we can find that the magnitude of the first term is larger than that of the second term by a factor of about N, and Fig. 6 presents a straightforward visualization. Thus, we take the zero points $\{x_i^{(\alpha)}\}_{i=0}^{N-1}$ of $\mathcal{Q}_N^{(\alpha)}(x)$ as the superconvergence points, and we have

$$
\big(f(x_i^{(\alpha)})-\mathcal{I}_N^\alpha f(x_i^{(\alpha)})\big)'=\frac{1}{2\pi i}\int_{\partial\mathcal{S}_\rho}\frac{\mathcal{Q}_{N+1}^{(\alpha)}(x_i^{(\alpha)})\cdot\alpha/\big(x_i^{(\alpha)}(1-x_i^{(\alpha)})\big)}{\big(y-z(x_i^{(\alpha)})\big)^2}\left(\frac{v(y)}{H_{N+1}(y)}\right)\mathrm{d}y,
$$

and

$$
\begin{aligned}
&\big(f(x)-\mathcal{I}_N^\alpha f(x)\big)''\\
=&\frac{1}{2\pi i}\int_{\partial\mathcal{S}_\rho}\partial_x^2\left(\frac{H_{N+1}(z)}{y-z}\right)\frac{v(y)}{H_{N+1}(y)}\mathrm{d}y\\
=&\frac{1}{2\pi i}\int_{\partial\mathcal{S}_\rho}\Bigg(\frac{\big(H_{N+1}'(z)z'(x)\big)'}{y-z}+\frac{H_{N+1}'(z)\big(z'(x)\big)^2}{(y-z)^2}+\frac{\big(H_{N+1}(x)z'(x)\big)'}{(y-z)^2}\\
&\qquad+\frac{H_{N+1}(z)z'(x)-2(y-z)z'(x)}{(y-z)^4}\Bigg)\frac{v(y)}{H_{N+1}(y)}\mathrm{d}y\\
=&\frac{1}{2\pi i}\int_{\partial\mathcal{S}_\rho}\Bigg(\frac{2(N+1)2N\mathcal{Q}_{N-1}^{(\alpha)}(x)\alpha^2/(x-x^2)^2}{y-z}+\frac{2(N+1)\mathcal{Q}_N^{(\alpha)}(x)\alpha(2x-1)/(x-x^2)^2}{y-z}\\
&\qquad+\frac{2(N+1)\mathcal{Q}_N^{(\alpha)}(x)\alpha^2/(x-x^2)^2}{(y-z)^2}+\frac{\mathcal{Q}_{N+1}^{(\alpha)}(x)\alpha(2x-1)/(x-x^2)^2}{(y-z)^2}\\
&\qquad+\frac{2\mathcal{Q}_{N+1}^{(\alpha)}(x)\alpha^2/(x-x^2)^2}{(y-z)^3}\Bigg)\frac{v(y)}{H_{N+1}(y)}\mathrm{d}y.
\end{aligned}
\tag{3.5}
$$

Considering $|y-z|\leqslant\rho$, let

$$
f_1(x)=2(N+1)2N\mathcal{Q}_{N-1}^{(\alpha)}(x)\frac{\alpha^2}{(x-x^2)^2},
$$

$$f_2(x) = 2(N+1)\mathcal{Q}_N^{(\alpha)}(x)\frac{\alpha(2x-1)}{(x-x^2)^2} + 2(N+1)\mathcal{Q}_N^{(\alpha)}(x)\frac{\alpha^2}{(x-x^2)^2},$$
$$f_3(x) = \mathcal{Q}_{N+1}^{(\alpha)}(x)\frac{\alpha(2x-1)}{(x-x^2)^2} + 2\mathcal{Q}_{N+1}^{(\alpha)}(x)\frac{\alpha^2}{x^2(1-x)^2}.$$

We can find that the magnitude of $f_1(x)$ is the largest in (3.5) and Fig. 7 allows for a more intuitive understanding. Then we take the zero points $\{x_i^{(\alpha)}\}_{i=0}^{N-2}$ of $\mathcal{Q}_{N-1}^{(\alpha)}(x)$ as the superconvergence points, and we have

$$\begin{aligned}
&\left(f(x_i^{(\alpha)}) - \mathcal{I}_N^\alpha f(x_i^{(\alpha)})\right)'' \\
=&\frac{1}{2\pi i}\int_{\partial\mathcal{S}_\rho}\Bigg(\frac{2(N+1)\mathcal{Q}_N^{(\alpha)}(x_i^{(\alpha)})(2x_i^{(\alpha)}-1)/(x_i^{(\alpha)}-(x_i^{(\alpha)})^2)^2}{y-z} \\
&\qquad + \frac{2(N+1)\mathcal{Q}_N^{(\alpha)}(x_i^{(\alpha)})\alpha^2/(x_i^{(\alpha)}-(x_i^{(\alpha)})^2)^2}{(y-z)^2} \\
&\qquad + \frac{\mathcal{Q}_{N+1}^{(\alpha)}(x_i^{(\alpha)})(2x_i^{(\alpha)}-1)/(x_i^{(\alpha)}-(x_i^{(\alpha)})^2)^2}{(y-z)^2} \\
&\qquad + \frac{2\mathcal{Q}_{N+1}^{(\alpha)}(x_i^{(\alpha)})\alpha^2/((x_i^{(\alpha)})^2(1-x_i^{(\alpha)})^2)}{(y-z)^3}\Bigg)\frac{v(y)}{H_{N+1}(y)}\mathrm{d}y.
\end{aligned}$$

Theorem 3.1. *If f is analytic in the strip $\mathcal{T}_\rho$ for some $\rho>0$ and $|f(x)|\le\hat{\mathcal{K}}|x|^\sigma$ for some $\sigma\in\mathbb{R}$ as $|x|\to 0$ and 1 within the strip, then consider $\mathcal{I}_N^\alpha f(x)$ to be the interpolant of $f(x)$ defined in* (3.2)*. For $k\ll N$, the superconvergence points of $\partial_x^{k+1}(f(x)-\mathcal{I}_N^\alpha f(x))$ are the zero points $x_i^{(\alpha)}$ of $\mathcal{Q}_{N-k}^{(\alpha)}(x)$.*

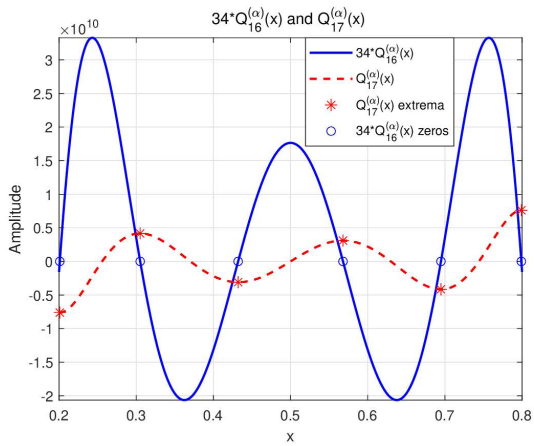

Figure 6: The magnitude of $34\mathcal{Q}_{16}^{(\alpha)}(x)$ and $\mathcal{Q}_{17}^{(\alpha)}(x)$ of $N=16$.

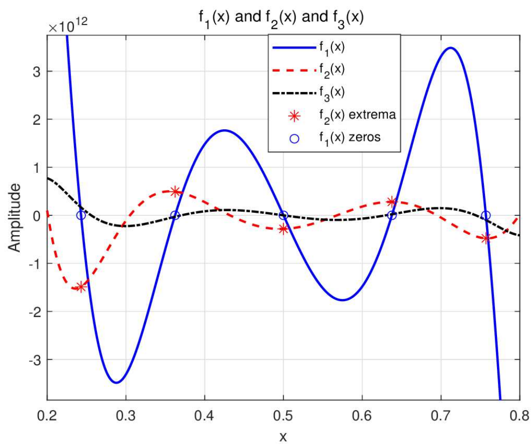

Figure 7: The magnitude of $f_1(x), f_2(x)$ and $f_3(x)$ of $N=16$.

Proof.

$$\partial_x^{k+1}\big(u(x)-\mathcal{I}_N^{\alpha}u(x)\big)=:\partial_x^{k+1}R_N(u;x).$$

And we denote that

$$v(z)=u\left(\frac{e^{z/\alpha}}{1+e^{z/\alpha}}\right),\quad v_N(z)=\mathcal{I}_N^h v(z).$$

Then $v(z)$ satisfies the conditions of Theorem 2.1. In view of (2.4), we have

$$\begin{aligned}\partial_x^{k+1}R_N(u;x)&=\partial_x^{k+1}\big(v(z)-\mathcal{I}_N^h v(z)\big)\\&=\frac{1}{2\pi i}\int_{\partial\mathcal{S}_\rho}\partial_x^{k+1}\left(\frac{H_{N+1}(z)}{y-z}\right)\left(\frac{v(y)}{H_{N+1}(y)}\right)\mathrm{d}y\\&=\frac{1}{2\pi i}\sum_{l=0}^{k+1}\binom{k+1}{l}\int_{\partial\mathcal{S}_\rho}\frac{\partial_x^l H_{N+1}(z)}{(y-z)^{k-l+2}}\frac{v(y)}{H_{N+1}(y)}\mathrm{d}y.\end{aligned}\tag{3.6}$$

Let $z=\alpha\log(x/(1-x))$, and we have

$$u(x)-\mathcal{I}_N^{\alpha}u(x)=v(z)-\mathcal{I}_N^h v(z)=:R_N^h(z),$$

which is a Hermite interpolation reminder. By the Faà di Bruno's formula

$$\begin{aligned}&(f\circ g)^{(n)}(x)\\=&\sum_{\substack{m_1,\dots,m_n\in\mathbb{N}\\1m_1+\cdots+nm_n=n}}\frac{n!}{m_1!1!^{m_1}\cdots m_n!n!^{m_n}}f^{(m_1+\cdots+m_n)}\big(g(x)\big)\big(g'(x)\big)^{m_1}\cdots\big(g^{(n)}(x)\big)^{m_n},\end{aligned}\tag{3.7}$$

we can obtain

$$\begin{aligned}&\partial_x^{k+1}\big(u(x)-\mathcal{I}_N^{\alpha}u(x)\big)\\&=\partial_x^{k+1}\big(v(z)-\mathcal{I}_N^h v(z)\big)\\&=\frac{1}{2\pi i}\sum_{l=0}^{k+1}\binom{k+1}{l}\int_{\partial\mathcal{S}_\rho}\sum\frac{l!}{m_1!1!^{m_1}\cdots m_l!l!^{m_l}}{H_{N+1}}^{(M_l)}\big(z(x)\big)\big(z'(x)\big)^{m_1}\cdots\big(z^{(l)}(x)\big)^{m_l}\\&\quad\times\frac{1}{(y-z)^{k-l+2}}\frac{v(y)}{H_{N+1}(y)}\mathrm{d}y\\&=\frac{1}{2\pi i}\sum_{l=0}^{k+1}\binom{k+1}{l}\int_{\partial\mathcal{S}_\rho}\sum\frac{l!}{m_1!1!^{m_1}\cdots m_l!l!^{m_l}}2(N+1)2N\cdots2(N+2-M_l)H_{N+1-M_l}\big(z(x)\big)\\&\quad\times\big(z'(x)\big)^{m_1}\cdots\big(z^{(l)}(x)\big)^{m_l}\frac{1}{(y-z)^{k-l+2}}\frac{v(y)}{H_{N+1}(y)}\mathrm{d}y\\&=\frac{1}{2\pi i}\sum_{l=0}^{k+1}\binom{k+1}{l}\int_{\partial\mathcal{S}_\rho}\sum\frac{l!}{m_1!1!^{m_1}\cdots m_l!l!^{m_l}}2(N+1)2N\cdots2(N+2-M_l)\mathcal{Q}_{N+1-M_l}^{(\alpha)}(x)\\&\quad\times\big(z'(x)\big)^{m_1}\cdots\big(z^{(l)}(x)\big)^{m_l}\frac{1}{(y-z)^{k-l+2}}\frac{v(y)}{H_{N+1}(y)}\mathrm{d}y,\end{aligned}\tag{3.8}$$

where $M_l = m_1+\cdots+m_l$ and $M_{k+1} \leqslant k+1$ and R_N^h is the reminder of the Hermite interpolation defined in (2.4). From Theorem 2.1, we know that the largest magnitude of $2(N+1)2N\cdots 2(N+2-M_l)H_{N+1-M_l}$, $l=0,\ldots,k+1$ is $2(N+1)2N\cdots 2(N+2-(k+1))\times H_{N+1-(k+1)}$ with $M_{k+1}=k+1$. Thus, we take the zero points $\{x_i^{(\alpha)}\}_{i=0}^{N-k-1}$ of $\mathcal{Q}_{N-k}^{(\alpha)}(x)$ as the superconvergence points, and we have

$$\begin{aligned}&\partial_x^{k+1}\big(u\big(x_i^{(\alpha)}\big)-\mathcal{I}_N^{\alpha}u\big(x_i^{(\alpha)}\big)\big)\\&=\frac{1}{2\pi i}\sum_{l=0}^{k+1}\binom{k+1}{l}\int_{\partial\mathcal{S}_\rho}\sum\frac{l!}{m_1!1!^{m_1}\cdots m_l!l!^{m_l}}2(N+1)2N\cdots 2(N+2-M_l)\\&\quad\times\mathcal{Q}_{N+1-M_l}^{(\alpha)}\big(x_i^{(\alpha)}\big)\big(z'\big(x_i^{(\alpha)}\big)\big)^{m_1}\cdots\big(z^{(l)}\big(x_i^{(\alpha)}\big)\big)^{m_l}\frac{1}{(y-z)^{k-l+2}}\frac{v(y)}{H_{N+1}(y)}\mathrm{d}y,\end{aligned}\tag{3.9}$$

where the term $\mathcal{Q}_{N-k}^{(\alpha)}(x_i^{(\alpha)})=0$ with $M_{k+1}=k+1$. □

Remark 3.1. Given that

$$z^{(l)}(x)=\left(\frac{\alpha}{x(1-x)}\right)^{(l-1)}=\sum_{i=0}^{l-1}\binom{l-1}{i}\frac{(-1)^i\alpha i!}{x^{i+1}}\frac{(-1)^{l-i}(l-1-i)!}{(1-x)^{l-i}},$$

it indicates that (3.8) and (3.9) exhibit oscillations at $x=0$ and $x=1$. Consequently, the superconvergence phenomena occurring in the middle of the interval is difficult to observe. To facilitate a better observation of this superconvergence phenomenon, we plot the derivative errors of the MHFs-interpolations within the interval $[0.2,0.8]$.

Next, we review the pseudo-derivative of f defined as $\widehat{\partial}_x f = x(1-x)\partial f$. The pseudo-derivative of $\mathcal{Q}_n^{(\alpha)}(x)$ satisfies

$$\widehat{\partial}_x\mathcal{Q}_n^{(\alpha)}(x)=\alpha\cdot 2n\mathcal{Q}_{n-1}^{(\alpha)}(x).\tag{3.10}$$

Then we derive the interpolation errors for the first and second pseudo-derivatives

$$\begin{aligned}&\widehat{\partial}_x\big(f(x)-\mathcal{I}_N^{\alpha}f(x)\big)\\&=\frac{1}{2\pi i}\int_{\partial\mathcal{S}_\rho}\widehat{\partial}_x\left(\frac{H_{N+1}(z(x))}{y-z}\right)\frac{v(y)}{H_{N+1}(y)}\mathrm{d}y\\&=\frac{1}{2\pi i}\int_{\partial\mathcal{S}_\rho}\left(\frac{2\alpha(N+1)\mathcal{Q}_N^{(\alpha)}(x)}{y-z}+\frac{\alpha\mathcal{Q}_{N+1}^{(\alpha)}(x)}{(y-z)^2}\right)\frac{v(y)}{H_{N+1}(y)}\mathrm{d}y,\end{aligned}\tag{3.11}$$

and

$$\begin{aligned}&\widehat{\partial}_x^2\big(f(x)-\mathcal{I}_N^{\alpha}f(x)\big)\\&=\frac{1}{2\pi i}\int_{\partial\mathcal{S}_\rho}\left(\frac{4\alpha^2(N+1)N\mathcal{Q}_{N-1}^{(\alpha)}(x)}{y-z}+\frac{4\alpha^2(N+1)\mathcal{Q}_N^{(\alpha)}(x)}{(y-z)^2}+\frac{\alpha^2\mathcal{Q}_{N+1}^{(\alpha)}(x)}{(y-z)^3}\right)\frac{v(y)}{H_{N+1}(y)}\mathrm{d}y.\end{aligned}\tag{3.12}$$

Similarly with (2.5) and (2.6), we take the zeros of $\mathcal{Q}_N^{(\alpha)}(x)$ and $\mathcal{Q}_{N-1}^{(\alpha)}(x)$ as the superconvergence points of (3.11) and (3.12), respectively. Comparing (3.11) and (3.12) with (3.4) and (3.5). The superconvergence of $\widehat{\partial}_x^k(f(x)-\mathcal{I}_N^\alpha f(x))$ is better and "cleaner" than that of $\partial_x^k(f(x)-\mathcal{I}_N^\alpha f(x))$. From (3.10), we have

$$\widehat{\partial}_x^k \mathcal{Q}_n^{(\alpha)}(x)=\alpha^k 2n2(n-1)\cdots 2(n-k+1)\mathcal{Q}_{n-k}^{(\alpha)}(x).$$

Then by a similar derivation to Theorem 2.1, we can obtain

Theorem 3.2. *If f is analytic in the strip $\mathcal{T}_\rho$ for some $\rho>0$ and $|f(x)|\le\hat{\mathcal{K}}|x|^\sigma$ for some $\sigma\in\mathbb{R}$ as $|x|\to 0$ and 1 within the strip, then consider $\mathcal{I}_N^\alpha f(x)$ to be the interpolant of $f(x)$ defined in* (3.2)*. For $k\ll N$, the superconvergence points of $\widehat{\partial}_x^{k+1}(f(x)-\mathcal{I}_N^\alpha f(x))$ are the zero points $x_i^{(\alpha)}$ of $\mathcal{Q}_{N-k}^{(\alpha)}(x)$.*

4 Generalized mapped Hermite functions

The MHFs are capable of resolving certain type of singularities at $t=0$ and $t=1$, but the term $\mathcal{Q}_n^{(\alpha)}(x)$ in MHFs involving $\{\log(x/(1-x))^k\}_{k=0}^n$ grows very fast near t=0 and t=1. This behavior may affect the accuracy in many situations. Therefore, we develop the GMHFs, which are more suitable for numerical approximations of functions with weak singularities at the two endpoints.

Definition 4.1. *Let $\alpha,\beta\ge 0$, the GMHFs can be defined by*

$$\mathcal{Q}_n^{(\alpha,\frac{\beta}{2})}(x)=x^{\frac{\beta}{2}}(1-x)^{\frac{\beta}{2}}H_n(z(x))=x^{\frac{\beta}{2}}(1-x)^{\frac{\beta}{2}}H_n\left(\alpha\log\left(\frac{x}{1-x}\right)\right),\quad n=0,1,\ldots. \tag{4.1}$$

In particular, $\mathcal{Q}_n^{(\alpha,0)}(x)=\mathcal{Q}_n^{(\alpha)}(x)$.

Next, we develop the basic properties of the GMHFs by the following proposition.

Proposition 4.1. *The GMHFs have the following properties:*

(1) Three-term recurrence relation

$$\begin{cases}\mathcal{Q}_{n+1}^{(\alpha,\frac{\beta}{2})}(x)=2\alpha\log\left(\dfrac{x}{1-x}\right)\mathcal{Q}_n^{(\alpha,\frac{\beta}{2})}(x)-2n\mathcal{Q}_{n-1}^{(\alpha,\frac{\beta}{2})}(x),\quad n\ge 1,\\ \mathcal{Q}_0^{(\alpha,\frac{\beta}{2})}(x)=x^{-\frac{\beta}{2}}(1-x)^{-\frac{\beta}{2}},\\ \mathcal{Q}_1^{(\alpha,\frac{\beta}{2})}(x)=2\alpha\log\left(\dfrac{x}{1-x}\right)x^{-\frac{\beta}{2}}(1-x)^{-\frac{\beta}{2}}.\end{cases} \tag{4.2}$$

(2) Derivative relation

$$\partial_x\mathcal{Q}_n^{(\alpha,\frac{\beta}{2})}(x)=\frac{\beta}{2}(1-2x)\mathcal{Q}_n^{(\alpha,\frac{\beta}{2}-1)}(x)+\lambda_n\alpha\mathcal{Q}_{n-1}^{(\alpha,\frac{\beta}{2}-1)}(x), \tag{4.3}$$

where λ_n grows linearly with respect to n, to be precise, $\lambda_n=2n$.

(3) Orthogonality

$$\int_0^1 \mathcal{Q}_n^{(\alpha,\frac{\beta}{2})}(x)\mathcal{Q}_m^{(\alpha,\frac{\beta}{2})}(x)\chi^{\alpha,\beta}(x)dx=\gamma_n^{(\alpha)}\delta_{mn} \tag{4.4}$$

with

$$\chi^{\alpha,\beta}(x)=\frac{e^{-\alpha^2\log^2(x/(1-x))}}{x^{1+\beta}(1-x)^{1+\beta}},\quad \gamma_n^{(\alpha)}=\frac{\sqrt{\pi}2^n n!}{\alpha}. \tag{4.5}$$

(4) Sturm-Liouville problem

$$\begin{aligned}&\frac{1}{\alpha^2}x^{1+\frac{\beta}{2}}(1-x)^{1+\frac{\beta}{2}}\partial_x\Big(x(1-x)\partial_x\Big(x^{-\frac{\beta}{2}}(1-x)^{-\frac{\beta}{2}}\mathcal{Q}_n^{(\alpha,\frac{\beta}{2})}(x)\Big)\Big)\\&\quad+2x(1-x)\log\left(\frac{x}{1-x}\right)\mathcal{Q}_n^{(\alpha,\frac{\beta}{2})}(x)+\lambda_n\mathcal{Q}_n^{(\alpha,\frac{\beta}{2})}(x)=0.\end{aligned} \tag{4.6}$$

(5) GMHFs-Gauss quadrature. Let $\{x_j^{(\alpha)},\chi_j^{\alpha}\}_{j=0}^N$ *be the Gauss nodes and weights of* $Q_{N+1}^{(\alpha)}(z)$*. Denote*

$$x_j^{(\alpha,\beta)}=x_j^{(\alpha)},\quad \chi_j^{\alpha,\beta}=\frac{\chi_j^{\alpha}}{x_j^{(\alpha,\beta)}\big(1-x_j^{(\alpha,\beta)}\big)}. \tag{4.7}$$

Then

$$\int_0^1 f(x)\chi^{\alpha,\beta}(x)dx=\sum_{j=0}^N f\big(x_j^{(\alpha,\beta)}\big)\cdot\chi_j^{\alpha,\beta},\quad \forall f\in\mathcal{P}_{2N+1}^{\beta,\log}, \tag{4.8}$$

where

$$\mathcal{P}_k^{\beta,\log}:=\big\{x^\beta(1-x)^\beta p(x): p(x)\in\mathcal{P}_k^{\log}\big\}$$

with

$$\mathcal{P}_k^{\log}:=\mathrm{span}\left\{1,\log\left(\frac{x}{(1-x)}\right),\log^2\left(\frac{x}{1-x}\right),\ldots,\log^k\left(\frac{x}{1-x}\right)\right\}.$$

Proof. We plug (4.1) into (A.1) and can obtain (4.2). For (4.3), in view of (A.2), we have

$$\begin{aligned}\partial_x\mathcal{Q}_n^{(\alpha,\frac{\beta}{2})}(x)&=\frac{\beta}{2}x^{\frac{\beta}{2}-1}(1-x)^{\frac{\beta}{2}-1}(1-2x)\mathcal{Q}_n^{(\alpha)}(x)+x^{\frac{\beta}{2}}(1-x)^{\frac{\beta}{2}}\partial_x\mathcal{Q}_n^{(\alpha)}(x)\\&=\frac{\beta}{2}(1-2x)\mathcal{Q}_n^{(\alpha,\frac{\beta}{2}-1)}(x)+\lambda_n\alpha\mathcal{Q}_{n-1}^{(\alpha,\frac{\beta}{2}-1)}(x).\end{aligned}$$

For (4.4), we have

$$\begin{aligned}&\int_0^1\mathcal{Q}_n^{(\alpha,\frac{\beta}{2})}(x)\mathcal{Q}_m^{(\alpha,\frac{\beta}{2})}(x)\chi^{\alpha,\beta}(x)dx\\&=\int_0^1 x^\beta(1-x)^\beta\mathcal{Q}_n^{(\alpha)}(x)\mathcal{Q}_m^{(\alpha)}(x)\frac{e^{-\alpha^2\log^2(x/(1-x))}}{x^{1+\beta}(1-x)^{1+\beta}}dx\\&=\int_0^1\mathcal{Q}_n^{(\alpha)}(x)\mathcal{Q}_m^{(\alpha)}(x)\chi^\alpha(x)dx=\gamma_n^{(\alpha)}\delta_{mn}.\end{aligned}$$

We plug

$$\mathcal{Q}_n^{(\alpha)}(x)=x^{-\frac{\beta}{2}}(1-x)^{-\frac{\beta}{2}}\mathcal{Q}_n^{(\alpha,\frac{\beta}{2})}(x)$$

into (A.6) and then we obtain (4.6). For (4.8), let $x=e^{z/\alpha}/(1+e^{z/\alpha})$ and we can obtain

$$\begin{aligned}\int_0^1 f(x)\chi^{(\alpha,\beta)}dx&=\int_{-\infty}^{\infty}f\left(\frac{e^{z/\alpha}}{1+e^{z/\alpha}}\right)\chi^{(\alpha,\beta)}\left(\frac{e^{z/\alpha}}{1+e^{z/\alpha}}\right)\frac{1}{\alpha}\frac{e^{z/\alpha}}{(1+e^{z/\alpha})^2}dz\\&=\int_{-\infty}^{\infty}f\left(\frac{e^{z/\alpha}}{1+e^{z/\alpha}}\right)e^{-z^2}\frac{(1+e^{z/\alpha})^{2\beta}}{\alpha(e^{z/\alpha})^{\beta}}dz\\&=\sum_{j=0}^{N}f\big(x_j^{(\alpha,\beta)}\big)\cdot\frac{\chi_j^{\alpha}}{\big(x_j^{(\alpha,\beta)}\big)^{\beta}\big(1-x_j^{(\alpha,\beta)}\big)^{\beta}}\end{aligned}$$

with

$$\chi^{(\alpha,\beta)}=\frac{e^{-\alpha^2\log^2(x/(1-x))}}{x^{1+\beta}(1-x)^{1+\beta}}.$$

The proof is complete. □

4.1 Projection estimate

For $\beta\in\mathbb{R}$, the projection operator $\Pi_N^{(\alpha,\beta)}:\ L^2_{\chi^{\alpha,\beta}}(I)\to\mathcal{P}_k^{\beta,\log}$ is defined by

$$\big(u-\Pi_N^{(\alpha,\beta)}u,v\big)_{\chi^{\alpha,\beta}}=0,\quad u\in L^2_{\chi^{\alpha,\beta}}(I),\quad v\in\mathcal{P}_N^{\beta,\log},$$

or equivalently,

$$\Pi_N^{(\alpha,\beta)}u=\sum_{n=0}^{N}\widehat{u}_n^{\alpha,\beta}\mathcal{Q}_n^{(\alpha,\frac{\beta}{2})},\quad \widehat{u}_n^{\alpha,\beta}=\big(\gamma_n^{(\alpha)}\big)^{-1}\int_I u(x)\mathcal{Q}_n^{(\alpha,\frac{\beta}{2})}(x)\chi^{\alpha,\beta}(x)dx,$$

where $\chi^{\alpha,\beta}$ is the weight function defined in (4.5) and $L^2_{\chi^{\alpha,\beta}}(I)$ is the L^2-weighted space defined as

$$L^2_{\chi^{\alpha,\beta}}(I)=\left\{u:\int_I|u(x)|^2\chi^{\alpha,\beta}dx<\infty\right\}.$$

For any $u\in L^2_{\chi^{\alpha,\beta}}(I)$, the pseudo-derivative of u is defined as

$$\widehat{\partial}_{\frac{\beta}{2},x}u=x^{1+\frac{\beta}{2}}(1-x)^{1+\frac{\beta}{2}}\partial_x\Big(x^{-\frac{\beta}{2}}(1-x)^{-\frac{\beta}{2}}u\Big).\tag{4.9}$$

In particular, when $\beta=0$, we have $\widehat{\partial}_{0,x}u=x(1-x)\partial_x u$.

By (A.2), we have

$$\begin{aligned}\widehat{\partial}_{\frac{\beta}{2},x}\mathcal{Q}_n^{(\alpha,\frac{\beta}{2})} &= x^{1+\frac{\beta}{2}}(1-x)^{1+\frac{\beta}{2}}\partial_x\Big(x^{-\frac{\beta}{2}}(1-x)^{-\frac{\beta}{2}}x^{\frac{\beta}{2}}(1-x)^{\frac{\beta}{2}}\mathcal{Q}_n^{(\alpha)}(x)\Big)\\ &= x^{1+\frac{\beta}{2}}(1-x)^{1+\frac{\beta}{2}}\partial_x\mathcal{Q}_n^{(\alpha)}(x) = x^{1+\frac{\beta}{2}}(1-x)^{1+\frac{\beta}{2}}\lambda_n\alpha\frac{\mathcal{Q}_{n-1}^{(\alpha)}(x)}{x(1-x)}\\ &= \lambda_n\alpha x^{\frac{\beta}{2}}(1-x)^{\frac{\beta}{2}}\mathcal{Q}_{n-1}^{(\alpha)}(x) = \alpha\lambda_n\mathcal{Q}_{n-1}^{(\alpha,\frac{\beta}{2})}(x). \end{aligned} \tag{4.10}$$

Next, we define the non-uniformly weighted Sobolev spaces

$$A_{\alpha,\beta}^k(I) = \Big\{v\in L^2_{\chi^{\alpha,\beta}}(I) : \widehat{\partial}_{\frac{\beta}{2},x}v \in L^2_{\chi^{\alpha+j,\beta}}(I), j=1,2,\ldots,k\Big\}, \quad k\in\mathbb{N}=\{1,2,\ldots\}$$

with the semi-norm and norm defined by

$$|v|_{A^m_{\alpha,\beta}} := \Big\|\widehat{\partial}^m_{\frac{\beta}{2},x}v\Big\|_{\chi^{\alpha+m,\beta}}, \quad \|v\|_{A^m_{\alpha,\beta}} := \left(\sum_{k=0}^m |v|^2_{A^k_{\alpha,\beta}}\right)^{\frac{1}{2}}.$$

Then we present the result of the projection error estimate.

Theorem 4.1. *For any* $u\in A^m_{\alpha,\beta}(I)$*, and* $0\leqslant k\leqslant \tilde{m}=\min\{m,N+1\}$*, we have*

$$\Big\|\widehat{\partial}^k_{\frac{\beta}{2},x}\big(u-\Pi_N^{(\alpha,\beta)}u\big)\Big\|_{\chi^{\alpha+k,\beta}} \leqslant 2^{\frac{k-\tilde{m}}{2}}\alpha^{k-1\tilde{m}}N^{\frac{-\tilde{m}+k}{2}}\big\|\widehat{\partial}^{\tilde{m}}_x u\big\|_{\chi^{\alpha+\tilde{m},\beta}}.$$

Proof. In view of (4.10),

$$\widehat{\partial}_{\frac{\beta}{2},x}\mathcal{Q}_n^{(\alpha,\frac{\beta}{2})} = 2n\alpha\mathcal{Q}_{n-1}^{(\alpha,\frac{\beta}{2})}(x).$$

Thus,

$$\widehat{\partial}^k_x\mathcal{Q}_n^{(\alpha,\frac{\beta}{2})}(x) = \frac{(2\alpha)^k n!}{(n-k)!}\mathcal{Q}_{n-k}^{(\alpha,\beta/2)}(x).$$

For any $u\in A^m_{\alpha,\beta}(I)$,

$$\begin{aligned} u(x) &= \sum_{n=0}^{\infty}\hat{u}_n^{\alpha}\mathcal{Q}_n^{(\alpha,\frac{\beta}{2})}(x),\\ \|\widehat{\partial}^k_x u\|^2_{\chi^{\alpha+k,\beta}} &= \sum_{n=k}^{\infty}\frac{(2\alpha)^{2k}(n!)^2}{\big((n-k)!\big)^2}\gamma_{n-k}^{(\alpha)}|\hat{u}_n^{\alpha}|^2. \end{aligned}$$

Then we have

$$\begin{aligned} &\big\|\widehat{\partial}^k_x\big(u-\Pi_N^{(\alpha,\beta)}u\big)\big\|^2_{\chi^{\alpha+k,\beta}}\\ =&\left\|\widehat{\partial}^k_x\sum_{n=N+1}^{\infty}\hat{u}_n^{\alpha}\mathcal{Q}_n^{(\alpha,\frac{\beta}{2})}(x)\right\|^2_{\chi^{\alpha+k,\beta}} \end{aligned}$$

$$
\begin{aligned}
&=\sum_{n=N+1}^{\infty}\frac{(2\alpha)^2k(n!)^2}{\big((n-k)!\big)^2}\gamma_{n-k}^{(\alpha)}|\hat{u}_n^{\alpha}|^2\\
&\leqslant\sum_{n=N+1}^{\infty}\frac{(2\alpha)^2k(n!)^2}{\big((n-k)!\big)^2}\gamma_{n-k}^{(\alpha)}\left(\frac{(2\alpha)^2\tilde{m}(n!)^2}{\big((n-k)!\big)^2}\gamma_{n-\tilde{m}}^{(\alpha)}\right)^{-1}\cdot\frac{(2\alpha)^2\tilde{m}(n!)^2}{\big((n-k)!\big)^2}\gamma_{n-\tilde{m}}^{(\alpha)}|\hat{u}_n^{\alpha}|^2\\
&\leqslant(2\alpha)^{2k-2\tilde{m}}\max_{N+1\leqslant n<\infty}\left(\frac{n!}{(n-k)!}\right)^2\gamma_{n-k}^{(\alpha)}\left(\left(\frac{n!}{(n-\tilde{m})}\right)^2\gamma_{n-\tilde{m}}^{(\alpha)}\right)^{-1}\sum_{n=N+1}^{\infty}\frac{(2\alpha)^2\tilde{m}(n!)^2}{\big((n-k)!\big)^2}\gamma_{n-\tilde{m}}^{(\alpha)}|\hat{u}_n^{\alpha}|^2\\
&=(2\alpha)^{2k-2\tilde{m}}\max_{N+1\leqslant n<\infty}\left(\frac{n!}{(n-k)!}\right)^2\gamma_{n-k}^{(\alpha)}\left(\left(\frac{n!}{(n-\tilde{m})!}\right)^2\gamma_{n-\tilde{m}}^{(\alpha)}\right)^{-1}\big\|\widehat{\partial}_x^{\tilde{m}}u\big\|^2_{\chi^{\alpha+\tilde{m},\beta}}\\
&=(2\alpha)^{2k-2\tilde{m}}\frac{(N+1-\tilde{m})!}{(N+1-k)!}2^{\tilde{m}-k}\big\|\widehat{\partial}_x^{\tilde{m}}u\big\|^2_{\chi^{\alpha+\tilde{m},\beta}},
\end{aligned}
$$

where the last equation is because

$$
\begin{aligned}
&\left(\frac{n!}{(n-k)!}\right)^2\gamma_{n-k}^{(\alpha)}\left(\left(\frac{n!}{(n-\tilde{m})!}\right)^2\gamma_{n-\tilde{m}}^{(\alpha)}\right)^{-1}\\
=&\left(\frac{(n-\tilde{m})!}{(n-k)!}\right)^2\cdot\frac{2^{n-k}(n-k)!}{2^{n-\tilde{m}}(n-\tilde{m})!}=\frac{(n-\tilde{m})!}{(n-k)!}\cdot2^{\tilde{m}-k}
\end{aligned}
$$

is monotonically decreasing with n increasing. Hence, we can obtain

$$
\big\|\widehat{\partial}_x^k\big(u-\Pi_N^{(\alpha,\beta)}u\big)\big\|^2_{\chi^{\alpha+k,\beta}}\leqslant2^{k-\tilde{m}}\alpha^{2k-2\tilde{m}}N^{-\tilde{m}+k}\big\|\widehat{\partial}_x^{\tilde{m}}u\big\|^2_{\chi^{\alpha+\tilde{m},\beta}}.
$$

The proof is complete. □

In Fig. 8, we depict the projection errors obtained using GMHFs and MHFs for the function $f(x)=x^{1/3}(1-x)^{1/3}$, which features two weakly singular points at $x=0$ and $x=1$, respectively. The results are presented for a fixed degree of basis, $N=28$. The parameters are uniformly taken as $\alpha=0.8$ and $\beta=1$. We observe that the projection error using GMHFs is uniformly small across the interval [0,1], whereas the error using MHFs is significantly larger than the machine precision. It means that the GMHFs-projection is more efficient than MHFs-projection.

4.2 Interpolation error

In this subsection, we will introduce the interpolation operator and the interpolation error by the GMHFs. For a smooth function $v\in C(I)$, the interpolation operator

$$
\mathcal{I}_N^{(\alpha,\beta)}:C(I)\ \rightarrow\ \mathcal{P}_k^{\beta,\log}(I)
$$

satisfies

$$
\mathcal{I}_N^{(\alpha,\beta)}f\big(x_j^{(\alpha,\beta)}\big)=f\big(x_j^{(\alpha,\beta)}\big),\quad j=0,1,\dots,N, \tag{4.11}
$$

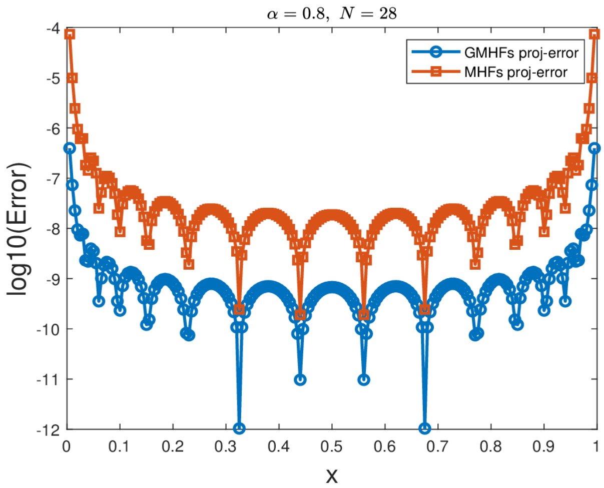

Figure 8: Projection error.

where $\mathcal{I}_N^{(\alpha,\beta)} f$ is the unique polynomial of degree N which interpolates $f(x)$ at the zero points $\{x_k\}_{k=0}^N$ of GMHFs $\mathcal{Q}_{N+1}^{(\alpha,\beta/2)}(x)$.

Next we present the estimate for the GMHFs-interpolation error.

Theorem 4.2. *Let m and N be positive integers, $\tilde{m}=\min\{m,N+1\}$. For any $v\in C(I)\cap A^m_{\alpha,\beta}(I)$ and $\widehat{\partial}_{\beta/2,x} v\in A^{m-1}_{\alpha,\beta}(I)$, we have*

$$
\begin{aligned}
&\big\|v-\mathcal{I}_N^{(\alpha,\beta)} v\big\|_{\chi^{\alpha,\beta}}\\
\leqslant& C(\sqrt{2}\alpha)^{(-\tilde{m})} N^{\frac{1}{6}-\frac{\tilde{m}}{2}} \left(\sqrt{\alpha}+2\max\left\{1,\frac{1}{\alpha}\right\} N^{-\frac{1}{2}}+\sqrt{2}\alpha\max\left\{1,\frac{1}{\alpha}\right\}+N^{-\frac{1}{6}}\right) \big\|\widehat{\partial}_x^{\tilde{m}} u\big\|_{\chi^{\alpha+\tilde{m},\beta}}.
\end{aligned}
$$

Proof. First, we have

$$
\big\|v-\mathcal{I}_N^{(\alpha,\beta)} v\big\|_{\chi^{\alpha,\beta}} = \left\|\mathcal{I}_N^{\alpha}\left(x^{\frac{\beta}{2}}(1-x)^{\frac{\beta}{2}} v\right) - x^{\frac{\beta}{2}}(1-x)^{\frac{\beta}{2}} v\right\|_{\chi^{\alpha}}.
$$

and

$$
\begin{aligned}
&\widehat{\partial}_x\left(x^{-\frac{\beta}{2}}(1-x)^{-\frac{\beta}{2}} v\right)\\
=& x^{-\frac{\beta}{2}}(1-x)^{-\frac{\beta}{2}} \widehat{\partial}_{\frac{\beta}{2},x} v,\ \widehat{\partial}_x^{\tilde{m}}\left(x^{-\frac{\beta}{2}}(1-x)^{-\frac{\beta}{2}} v\right)\\
=& x^{-\frac{\beta}{2}}(1-x)^{-\frac{\beta}{2}} \widehat{\partial}^{\tilde{m}}_{\frac{\beta}{2},x} v.
\end{aligned}
$$

Put the above results into [37, Theorem 2.4] and we can complete the proof. □

4.3 The superconvergence

First, we have

$$f(x)-\mathcal{I}_N^{(\alpha,\beta)}f(x)=x^{\frac{\beta}{2}}(1-x)^{\frac{\beta}{2}}\frac{1}{2\pi i}\int_{\partial\mathcal{S}_\rho}\frac{H_{N+1}\big(z(x)\big)}{y-z}\frac{x(y)^{-\beta/2}\big(1-x(y)\big)^{-\beta/2}v(y)}{H_{N+1}(y)}\mathrm{d}y.$$

Then we have

$$\begin{aligned}&\big(f(x)-\mathcal{I}_N^{(\alpha,\beta)}f(x)\big)'\\&=\left(\frac{\beta}{2}x^{\frac{\beta}{2}-1}(1-x)^{\frac{\beta}{2}}-\frac{\beta}{2}x^{\frac{\beta}{2}}(1-x)^{\frac{\beta}{2}-1}\right)\frac{1}{2\pi i}\int_{\partial\mathcal{S}_\rho}\frac{H_{N+1}\big(z(x)\big)}{y-z}\frac{x(y)^{-\beta/2}\big(1-x(y)\big)^{-\beta/2}v(y)}{H_{N+1}(y)}\mathrm{d}y\\&\quad+\frac{x^{\beta/2}(1-x)^{\beta/2}}{2\pi i}\int_{\partial\mathcal{S}_\rho}\partial_x\left(\frac{H_{N+1}\big(z(x)\big)}{y-z(x)}\right)\frac{x(y)^{-\beta/2}\big(1-x(y)\big)^{-\beta/2}v(y)}{H_{N+1}(y)}\mathrm{d}y\\&=\frac{1}{2\pi i}\int_{\partial\mathcal{S}_\rho}\left(\frac{(\beta/2)\big(1/x-1/(1-x)\big)\mathcal{Q}_{N+1}^{(\alpha,\beta/2)}(x)}{y-z}+\frac{2(N+1)\big(\alpha/\big(x(1-x)\big)\big)\mathcal{Q}_N^{(\alpha,\beta/2)}(x)}{y-z}\right.\\&\qquad\left.+\frac{\big(\alpha/\big(x(1-x)\big)\big)\mathcal{Q}_{N+1}^{(\alpha,\beta/2)}(x)}{(y-z)^2}\right)\frac{x(y)^{-\beta/2}\big(1-x(y)\big)^{-\beta/2}v(y)}{H_{N+1}(y)}\mathrm{d}y.\end{aligned}\tag{4.12}$$

Let

$$\begin{aligned}f_1(x)&=\left(\frac{\beta}{2}\left(\frac{1}{x}-\frac{1}{1-x}\right)+\frac{\alpha}{x(1-x)}\right)\mathcal{Q}_{N+1}^{(\alpha,\frac{\beta}{2})}(x),\\f_2(x)&=2(N+1)\frac{\alpha}{x(1-x)}\mathcal{Q}_N^{(\alpha,\frac{\beta}{2})}(x).\end{aligned}$$

Eq. (4.12) shows that the magnitude of $f_2(x)$ is larger than that of $f_1(x)$ by a factor of about N, and Fig. 9 provides a clearer visualization. Then we take the zero points $\{x_i^{(\alpha,\beta)}\}_{i=0}^{N-1}$ of $\mathcal{Q}_N^{(\alpha,\beta/2)}(x)$ as the superconvergence points, and we have

$$\begin{aligned}&\big(f(x_i^{(\alpha,\beta)})-\mathcal{I}_N^{(\alpha,\beta)}f(x_i^{(\alpha,\beta)})\big)'\\&=\frac{1}{2\pi i}\int_{\partial\mathcal{S}_\rho}\frac{f_1\big(x_i^{(\alpha,\beta)}\big)}{y-z\big(x_i^{(\alpha,\beta)}\big)}\frac{x(y)^{-\beta/2}\big(1-x(y)\big)^{-\beta/2}v(y)}{H_{N+1}(y)}\mathrm{d}y\\&\quad+\frac{1}{2\pi i}\int_{\partial\mathcal{S}_\rho}\frac{f_3\big(x_i^{(\alpha,\beta)}\big)}{y-z\big(x_i^{(\alpha,\beta)}\big)}\frac{x(y)^{-\beta/2}\big(1-x(y)\big)^{-\beta/2}v(y)}{H_{N+1}(y)}\mathrm{d}y,\end{aligned}$$

$$\begin{aligned}&\big(f(x)-\mathcal{I}_N^{(\alpha,\beta)}f(x)\big)''\\&=\frac{C_1(x)}{2\pi i}\int_{\partial\mathcal{S}_\rho}\frac{H_{N+1}\big(z(x)\big)}{y-z}\frac{x(y)^{-\beta/2}\big(1-x(y)\big)^{-\beta/2}v(y)}{H_{N+1}(y)}\mathrm{d}y\end{aligned}$$

$$
\begin{aligned}
&+\frac{C_2(x)}{2\pi i}\int_{\partial\mathcal{S}_\rho}\left(\frac{H_{N+1}\big(z(x)\big)}{y-z}\right)'\frac{x(y)^{-\beta/2}\big(1-x(y)\big)^{-\beta/2}v(y)}{H_{N+1}(y)}\mathrm{d}y\\
&+x^{\frac{\beta}{2}}(1-x)^{\frac{\beta}{2}}\frac{1}{2\pi i}\int_{\partial\mathcal{S}_\rho}\left(\frac{H_{N+1}\big(z(x)\big)}{y-z}\right)''\frac{x(y)^{-\beta/2}\big(1-x(y)\big)^{-\beta/2}v(y)}{H_{N+1}(y)}\mathrm{d}y\\
=&\frac{C_1(x)}{2\pi i}\int_{\partial\mathcal{S}_\rho}\frac{H_{N+1}\big(z(x)\big)}{y-z}\frac{x(y)^{-\beta/2}\big(1-x(y)\big)^{-\beta/2}v(y)}{H_{N+1}(y)}\mathrm{d}y\\
&+\frac{C_2(x)}{2\pi i}\int_{\partial\mathcal{S}_\rho}\left(\frac{2(N+1)H_N\big(z(x)\big)z'(x)}{y-z}+\frac{H_{N+1}\big(z(x)\big)z'(x)}{(y-z)^2}\right)\\
&\times\frac{x(y)^{-\beta/2}\big(1-x(y)\big)^{-\beta/2}v(y)}{H_{N+1}(y)}\mathrm{d}y\\
&+x^{\frac{\beta}{2}}(1-x)^{\frac{\beta}{2}}\frac{1}{2\pi i}\int_{\partial\mathcal{S}_\rho}\left(\frac{2(N+1)2NH_{N-1}\big(z(x)\big)\big(z'(x)\big)^2+2(N+1)H_N\big(z(x)\big)z''(x)}{y-z}\right.\\
&\qquad+\frac{4(N+1)H_N\big(z(x)\big)\big(z'(x)\big)^2+H_{N+1}\big(z(x)\big)z''(x)}{(y-z)^2}\\
&\qquad\left.+\frac{2H_{N+1}(z)\big(z'(x)\big)^2}{(y-z)^3}\right)\frac{x(y)^{-\beta/2}\big(1-x(y)\big)^{-\beta/2}v(y)}{H_{N+1}(y)}\mathrm{d}y\\
=&\frac{1}{2\pi i}\int_{\partial\mathcal{S}_\rho}\frac{C_3(x)\mathcal{Q}_{N+1}^{(\alpha,\beta/2)}(x)}{y-z}\mathrm{d}y\\
&+\beta\left(\frac{1}{x}-\frac{1}{1-x}\right)\left(\frac{2(N+1)\mathcal{Q}_N^{(\alpha,\beta/2)}(x)\alpha/\big(x(1-x)\big)}{(y-z)}+\frac{\mathcal{Q}_{N+1}^{(\alpha,\beta/2)}(x)\alpha/\big(x(1-x)\big)}{(y-z)^2}\right)\\
&+\left(\frac{2(N+1)2N\mathcal{Q}_{N-1}^{(\alpha,\beta/2)}(x)\alpha^2/(x-x^2)^2+2(N+1)\mathcal{Q}_N^{(\alpha,\beta/2)}(x)\alpha(2x-1)/(x-x^2)^2}{y-z}\right.\\
&\quad+\frac{4(N+1)\mathcal{Q}_N^{(\alpha,\beta/2)}(x)\alpha^2/(x-x^2)^2+\mathcal{Q}_{N+1}^{(\alpha,\beta/2)}(x)\alpha(2x-1)/(x-x^2)^2}{(y-z)^2}\\
&\quad\left.+\frac{2\mathcal{Q}_{N+1}^{(\alpha,\beta/2)}(x)\alpha^2/(x-x^2)^2}{(y-z)^3}\right)\frac{x(y)^{-\beta/2}\big(1-x(y)\big)^{-\beta/2}v(y)}{H_{N+1}(y)}\mathrm{d}y, \qquad (4.13)
\end{aligned}
$$

where

$$
\begin{aligned}
&C_1(x)=\frac{\beta}{2}\left(\frac{\beta}{2}-1\right)x^{\frac{\beta}{2}-2}(1-x)^{\frac{\beta}{2}}-\frac{\beta^2}{2}(x-x^2)^{\frac{\beta}{2}-1}+\frac{\beta}{2}\left(\frac{\beta}{2}-1\right)x^{\frac{\beta}{2}}(1-x)^{\frac{\beta}{2}-2},\\
&C_2(x)=\beta x^{\frac{\beta}{2}-1}(1-x)^{\frac{\beta}{2}}-\beta x^{\frac{\beta}{2}}(1-x)^{\frac{\beta}{2}-1},
\end{aligned}
$$

$$C_3(x)=\frac{\beta}{2}\left(\frac{\beta}{2}-1\right)x^{-2}-\frac{\beta^2}{2}(x-x^2)^{-1}+\frac{\beta}{2}\left(\frac{\beta}{2}-1\right)(1-x)^{-2}.$$

Let

$$\begin{aligned}
f_1(x)&=2(N+1)2N\frac{\alpha^2}{x^2(1-x)^2}\mathcal{Q}_{N-1}^{(\alpha,\frac{\beta}{2})}(x),\\
f_2(x)&=\left(2(N+1)\frac{\alpha}{x(1-x)}+2(N+1)\frac{\alpha(2x-1)}{(x-x^2)^2}+4(N+1)\frac{\alpha^2}{(x-x^2)^2}\right)\mathcal{Q}_{N}^{(\alpha,\frac{\beta}{2})}(x),\\
f_3(x)&=\left(1+\frac{1}{x(1-x)}+\frac{\alpha(2x-1)}{(x-x^2)^2}+2\frac{\alpha^2}{(x-x^2)^2}\right)\mathcal{Q}_{N+1}^{(\alpha,\frac{\beta}{2})}(x).
\end{aligned}$$

Eq. (4.13) indicates that the term $f_1(x)$ has the largest magnitude. Fig. 10 offers a clear visual representation. By considering the zero points $\{x_i^{(\alpha,\beta)}\}_{i=0}^{N-2}$ of $\mathcal{Q}_{N-1}^{(\alpha,\beta/2)}(x)$ as the superconvergence points, the following relationship holds:

$$\begin{aligned}
&\big(f\big(x_i^{(\alpha,\beta)}\big)-\mathcal{I}_N^{(\alpha,\beta)}f\big(x_i^{(\alpha,\beta)}\big)\big)''\\
=&\frac{C_3\big(x_i^{(\alpha,\beta)}\big)}{2\pi i}\int_{\partial\mathcal{S}_\rho}\frac{\mathcal{Q}_{N+1}^{(\alpha,\beta/2)}\big(x_i^{(\alpha,\beta)}\big)}{y-z(x_i^{(\alpha,\beta)})}+\beta\left(\frac{1}{x_i^{(\alpha,\beta)}}-\frac{1}{1-x_i^{(\alpha,\beta)}}\right)\\
&\times\left(\frac{2(N+1)\mathcal{Q}_N^{(\alpha,\beta/2)}\big(x_i^{(\alpha,\beta)}\big)\alpha/\big(x_i^{(\alpha,\beta)}\big(1-x_i^{(\alpha,\beta)}\big)\big)}{y-z\big(x_i^{(\alpha,\beta)}\big)}\right.\\
&\qquad\left.+\frac{\mathcal{Q}_{N+1}^{(\alpha,\beta/2)}\big(x_i^{(\alpha,\beta)}\big)\alpha/\big(x_i^{(\alpha,\beta)}\big(1-x_i^{(\alpha,\beta)}\big)\big)}{(y-z(x_i^{(\alpha,\beta)}))^2}\right)\\
&+\left(\frac{\mathcal{Q}_{N+1}^{(\alpha,\beta/2)}\big(x_i^{(\alpha,\beta)}\big)\alpha\big(2x_i^{(\alpha,\beta)}-1\big)/\big(x_i^{(\alpha,\beta)}-\big(x_i^{(\alpha,\beta)}\big)^2\big)^2}{\big(y-z\big(x_i^{(\alpha,\beta)}\big)\big)^2}\right.\\
&\qquad+\frac{2\mathcal{Q}_{N+1}^{(\alpha,\beta/2)}\big(x_i^{(\alpha,\beta)}\big)\alpha^2/\big(x_i^{(\alpha,\beta)}-\big(x_i^{(\alpha,\beta)}\big)^2\big)^2}{\big(y-z\big(x_i^{(\alpha,\beta)}\big)\big)^3}\\
&\qquad+\frac{2(N+1)\mathcal{Q}_N^{(\alpha,\beta/2)}\big(x_i^{(\alpha,\beta)}\big)\alpha\big(2x_i^{(\alpha,\beta)}-1\big)/\big(x_i^{(\alpha,\beta)}-\big(x_i^{(\alpha,\beta)}\big)^2\big)^2}{y-z\big(x_i^{(\alpha,\beta)}\big)}\\
&\qquad\left.+\frac{4(N+1)\mathcal{Q}_N^{(\alpha,\beta/2)}\big(x_i^{(\alpha,\beta)}\big)\alpha^2/\big(x_i^{(\alpha,\beta)}-\big(x_i^{(\alpha,\beta)}\big)^2\big)^2}{\big(y-z\big(x_i^{(\alpha,\beta)}\big)\big)^2}\right)\\
&\times\frac{x(y)^{-\beta/2}\big(1-x(y)\big)^{-\beta/2}v(y)}{H_{N+1}(y)}\mathrm{d}y.
\end{aligned}$$

Therefore, the following result can be obtained.

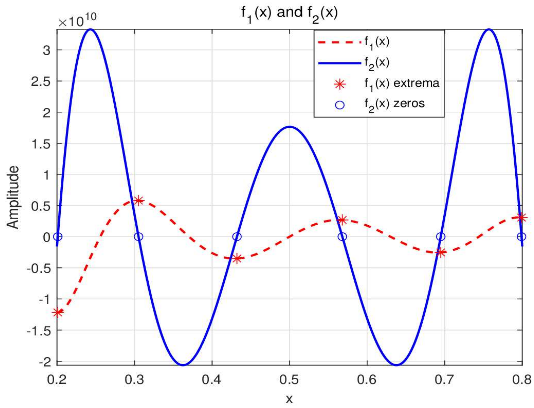

Figure 9: The magnitude of $f_1(x)$ and $f_2(x)$ and $f_3(x)$ of $N=16$.

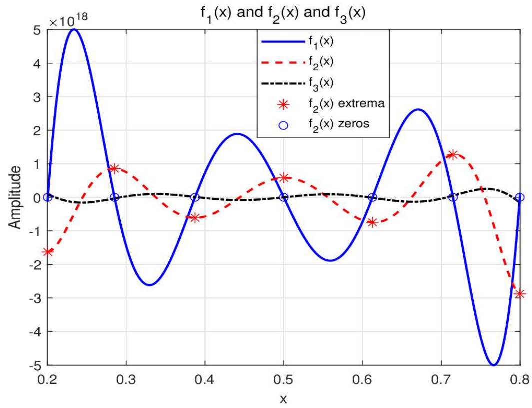

Figure 10: The magnitude of $f_1(x), f_2(x)$ and $f_3(x)$ of $N=24$.

Theorem 4.3. *If f is analytic in the strip $\mathcal{T}_\rho$ for some $\rho>0$ and $|f(x)|\le\hat{\mathcal{K}}|x|^\sigma$ for some $\sigma\in\mathbb{R}$ as $|x|\to 0$ and 1 within the strip, then consider $\mathcal{I}_N^{(\alpha,\beta)}f(x)$ to be the interpolant of $f(x)$ defined in* (4.11). *For $k\ll N$, the superconvergence points of $\partial_x^{k+1}(f(x)-\mathcal{I}_N^{(\alpha,\beta)}f(x))$ are the zero points $x_i^{(\alpha,\beta)}$ of $\mathcal{Q}_{N-k}^{(\alpha,\beta/2)}(x)$.*

Proof. For convenience, let $\mathcal{I}_N^{(\alpha,\beta)}u:=Q_N^{\alpha,\beta}\in\mathcal{P}_k^{\beta,\log}$. In view of (4.11), we see that $x^{-\beta/2}\times(1-x)^{-\beta/2}Q_N^{\alpha,\beta}$ is a function which interpolates $x^{-\beta/2}(1-x)^{-\beta/2}u$ at the points $\{x_j^{(\alpha,\beta)}\}_{j=0}^N$. Combining this observation with (3.6) and (4.1) enable us to obtain

$$\begin{aligned}
&x^{-\frac{\beta}{2}}(1-x)^{-\frac{\beta}{2}}u(x)-x^{-\frac{\beta}{2}}(1-x)^{-\frac{\beta}{2}}Q_N^{\alpha,\beta}(x)\\
=&x^{-\frac{\beta}{2}}(1-x)^{-\frac{\beta}{2}}u(x)-Q_N^{\alpha}(x)\\
=&x(z)^{-\frac{\beta}{2}}\big(1-x(z)\big)^{-\frac{\beta}{2}}v(z)-H_N(z)\\
=&\frac{1}{2\pi i}\int_{\partial\mathcal{S}_\rho}\frac{H_{N+1}(z)}{y-z}\frac{x(y)^{-\beta/2}\big(1-x(y)\big)^{-\beta/2}v(y)}{H_{N+1}(y)}dy,
\end{aligned}\tag{4.14}$$

where $x=e^{z/\alpha}/(1+e^{z/\alpha}),\quad z\in(-\infty,\infty)$. Then, we have

$$u(x)-Q_N^{\alpha,\beta}(x)=x^{\frac{\beta}{2}}(1-x)^{\frac{\beta}{2}}\frac{1}{2\pi i}\int_{\partial\mathcal{S}_\rho}\frac{H_{N+1}(z(x))}{y-z}\frac{x(y)^{-\beta/2}\big(1-x(y)\big)^{-\beta/2}v(y)}{H_{N+1}(y)}\mathrm{d}y.$$

Similarly with Theorem 3.1, from (3.7) we can obtain

$$\begin{aligned}
&\partial_x^{k+1}\big(u(x)-Q_N^{\alpha,\beta}(x)\big)\\
=&\partial_x^{k+1}\left(x^{\frac{\beta}{2}}(1-x)^{\frac{\beta}{2}}\frac{1}{2\pi i}\int_{\partial\mathcal{S}_\rho}\frac{H_{N+1}(z(x))}{y-z}\frac{x(y)^{-\beta/2}\big(1-x(y)\big)^{-\beta/2}v(y)}{H_{N+1}(y)}\mathrm{d}y\right)
\end{aligned}$$

$$
\begin{aligned}
&=\sum_{l=0}^{k+1}\binom{k+1}{l}\partial_x^l\left(\frac{1}{2\pi i}\int_{\partial\mathcal{S}_\rho}\frac{H_{N+1}\big(z(x)\big)}{y-z}\frac{x(y)^{-\beta/2}\big(1-x(y)\big)^{-\beta/2}v(y)}{H_{N+1}(y)}\mathrm{d}y\right)\partial_x^{k+1-l}\left(x^{\frac{\beta}{2}}(1-x)^{\frac{\beta}{2}}\right)\\
&=\sum_{l=0}^{k+1}\binom{k+1}{l}\left(\frac{1}{2\pi i}\int_{\partial\mathcal{S}_\rho}\partial_x^l\left(\frac{H_{N+1}\big(z(x)\big)}{y-z(x)}\right)\frac{x(y)^{-\beta/2}\big(1-x(y)\big)^{-\beta/2}v(y)}{H_{N+1}(y)}\mathrm{d}y\right)\\
&\quad\times\partial_x^{k+1-l}\big(x^{\frac{\beta}{2}}(1-x)^{\frac{\beta}{2}}\big)\\
&=\sum_{l=0}^{k+1}\binom{k+1}{l}\frac{1}{2\pi i}\sum_{j=0}^{l}\binom{l}{j}\int_{\partial\mathcal{S}_\rho}\sum\frac{l!}{m_1!1!^{m_1}\cdots m_j!j!^{m_j}}2(N+1)2N\cdots2(N+2-M_j)\\
&\quad\times\mathcal{Q}_{N+1-M_j}^{(\alpha)}(x)\big(z'(x)\big)^{m_1}\cdots\big(z^{(j)}(x)\big)^{m_j}\frac{1}{(y-z)^{l-j+1}}\frac{x(y)^{-\beta/2}\big(1-x(y)\big)^{-\beta/2}v(y)}{H_{N+1}(y)}\mathrm{d}y\\
&\quad\times\partial_x^{k+1-l}\big(x^{\frac{\beta}{2}}(1-x)^{\frac{\beta}{2}}\big),
\end{aligned}
$$

where $M_l=m_1+\cdots+m_l$ and $M_{k+1}\leqslant k+1$. From Theorem 3.1, we take the zero points $\{x_i^{(\alpha,\beta)}\}_{i=0}^{N-k-1}$ of the largest magnitude $\mathcal{Q}_{N-k}^{(\alpha,\beta/2)}(x)$, which is also the zero points of $H_{N+1-(k+1)}(z(x))$ with $M_{k+1}=k+1$ as the superconvergence points, and we have

$$
\begin{aligned}
&\partial_x^{k+1}\left(u\big(x_i^{(\alpha,\beta)}\big)-Q_N^{\alpha,\beta}\big(x_i^{(\alpha,\beta)}\big)\right)\\
&=\sum_{l=0}^{k+1}\binom{k+1}{l}\frac{1}{2\pi i}\sum_{j=0}^{l}\binom{l}{j}\int_{\partial\mathcal{S}_\rho}\sum\frac{l!}{m_1!1!^{m_1}\cdots m_j!j!^{m_j}}2(N+1)2N\cdots\\
&\quad\times2(N+2-M_j)\,\mathcal{Q}_{N+1-M_j}^{(\alpha)}\big(x_i^{(\alpha,\beta)}\big)\big(z'\big(x_i^{(\alpha,\beta)}\big)\big)^{m_1}\cdots\big(z^{(j)}\big(x_i^{(\alpha,\beta)}\big)\big)^{m_j}\frac{1}{(y-z)^{l-j+1}}\\
&\quad\times\frac{x(y)^{-\beta/2}\big(1-x(y)\big)^{-\beta/2}v(y)}{H_{N+1}(y)}\mathrm{d}y\cdot\partial_x^{k+1-l}\left(\big(x_i^{(\alpha,\beta)}\big)^{\frac{\beta}{2}}\big(1-x_i^{(\alpha,\beta)}\big)^{\frac{\beta}{2}}\right),
\end{aligned}
$$

where the term $2(N+1)2N\cdots2(N+1-k)\,\mathcal{Q}_{N-k}^{(\alpha)}(x_i^{(\alpha,\beta)})=0$ with $M_{k+1}=k+1$. □

Next, we derive the interpolation errors for the first and second pseudo-derivatives.

$$
\begin{aligned}
&\widehat{\partial}_{\frac{\beta}{2},x}\big(f(x)-\mathcal{I}_N^{(\alpha,\beta)}f(x)\big)\\
&=x^{1+\frac{\beta}{2}}(1-x)^{1+\frac{\beta}{2}}\partial_x\left(x^{-\frac{\beta}{2}}(1-x)^{-\frac{\beta}{2}}\big(f(x)-\mathcal{I}_N^{\alpha}f(x)\big)\right)\\
&=x^{1+\frac{\beta}{2}}(1-x)^{1+\frac{\beta}{2}}\frac{1}{2\pi i}\int_{\partial\mathcal{S}_\rho}\left(\frac{2\alpha(N+1)\,\mathcal{Q}_N^{(\alpha)}(x)\cdot\alpha/\big(x(1-x)\big)}{y-z}+\frac{\mathcal{Q}_{N+1}^{(\alpha)}(x)\cdot\alpha/\big(x(1-x)\big)}{(y-z)^2}\right)\\
&\quad\times\frac{x(y)^{-\beta/2}\big(1-x(y)\big)^{-\beta/2}v(y)}{H_{N+1}(y)}\mathrm{d}y
\end{aligned}
$$

$$=\frac{1}{2\pi i}\int_{\partial\mathcal{S}_\rho}\left(\frac{2\alpha(N+1)\mathcal{Q}_N^{(\alpha,\beta/2)}(x)}{y-z}+\frac{\mathcal{Q}_{N+1}^{(\alpha,\beta/2)}(x)}{(y-z)^2}\right)\frac{x(y)^{-\beta/2}\big(1-x(y)\big)^{-\beta/2}v(y)}{H_{N+1}(y)}\mathrm{d}y, \quad (4.15)$$

where $\widehat{\partial}_{\beta/2,x}f$ is defined in (4.9).

$$\begin{aligned}&\widehat{\partial}^{(2)}_{\frac{\beta}{2},x}\big(f(x)-\mathcal{I}_N^{(\alpha,\beta)}f(x)\big)\\&=\widehat{\partial}_{\frac{\beta}{2},x}\Big(\widehat{\partial}_{\frac{\beta}{2},x}\big(f(x)-\mathcal{I}_N^{\alpha}f(x)\big)\Big)\\&=\frac{1}{2\pi i}\int_{\partial\mathcal{S}_\rho}\left(\frac{4\alpha^2(N+1)N\mathcal{Q}_{N-1}^{(\alpha,\beta/2)}(x)}{y-z}+\frac{4\alpha^2(N+1)\mathcal{Q}_N^{(\alpha,\beta/2)}(x)}{(y-z)^2}+\frac{2\alpha\mathcal{Q}_{N+1}^{(\alpha,\beta/2)}(x)}{(y-z)^3}\right)\\&\qquad\times\frac{x(y)^{-\beta/2}\big(1-x(y)\big)^{-\beta/2}v(y)}{H_{N+1}(y)}\mathrm{d}y. \qquad (4.16)\end{aligned}$$

We consider the zeros of $\mathcal{Q}_N^{(\alpha,\beta/2)}(x)$ and $\mathcal{Q}_{N-1}^{(\alpha,\beta/2)}(x)$ as the superconvergence points for (4.15) and (4.16), respectively. By comparing (4.15) and (4.16) with (4.12) and (4.13), it is observed that the superconvergence of $\widehat{\partial}^{k+1}_{\beta/2,x}(f(x)-\mathcal{I}_N^{(\alpha,\beta)}f(x)),k=0,1$ is better and "cleaner" than that of $\partial^{k+1}_{\beta/2,x}(f(x)-\mathcal{I}_N^{(\alpha,\beta)}f(x))$. Consequently, following a derivation similar to that of Theorem 2.1, we have,

Theorem 4.4. *If f is analytic in the strip $\mathcal{T}_\rho$ for some $\rho>0$ and $|f(x)|\le\hat{\mathcal{K}}|x|^\sigma$ for some $\sigma\in\mathbb{R}$ as $|x|\to 0$ and 1 within the strip, then consider $\mathcal{I}_N^{(\alpha,\beta)}f(x)$ to be the interpolant of $f(x)$ defined in* (4.11). *For $k\ll N$, the superconvergence points of $\widehat{\partial}_x^{k+1}(f(x)-\mathcal{I}_N^{(\alpha,\beta)}f(x))$ are the zero points $x_i^{(\alpha)}$ of $\mathcal{Q}_{N-k}^{(\alpha)}(x)$.*

5 Numerical results and concluding remarks

In this section, we present some numerical examples to illustrate the superconvergence phenomenon by comparing the convergence behavior between the MHFs-interpolation superconvergence points and GMHFs-interpolation superconvergence points. Finally, we conclude the paper with some remarks.

5.1 Numerical results

Example 5.1. Consider the function

$$u(x)=-3\log\Big(2-\frac{x}{3}\Big)$$

with $\partial_x u(x)=-1/(2-x/3)$.

The first derivative error of Hermite spectral interpolant at the superconvergence points shown in Theorem 2.1 is depicted in Fig. 11.

Example 5.2. We consider the function

$$u(x)=\frac{1}{10}\text{atan}(10x-5), \quad x\in(0,1)$$

with $\partial_x u(x)=1/((10x-5)^2+1)$ which is the well-known Runge example.

The first-order derivative errors of the two types of nonpolynomial interpolants at the superconvergence points identified in Theorems 3.1 and 4.3 are depicted in Figs. 12 and 13, respectively. Furthermore, Figs. 14 and 15 illustrate the pseudo-derivative errors of these interpolants, with focus on the superconvergence points from Theorems 3.1 and 4.3. It is notable that the superconvergence of the pseudo-derivative errors is superior to that of the derivative errors.

Example 5.3. Then we consider the function

$$u(x)=-\frac{1}{2}\log(3-2x),$$

with $\partial_x u(x)=1/(3-2x)$.

We plot the first-order derivative errors and pseudo-derivative errors of the interpolants in Figs. 16-19. The first-order pseudo-derivative errors are smaller than the derivative errors, demonstrating faster convergence. Additionally, this function converges much more rapidly in comparison to Example 5.2.

5.2 Concluding remarks

In this paper, we analyze the superconvergence phenomenon and identify superconvergence points for derivatives of several interpolants based on Hermite polynomials/func-

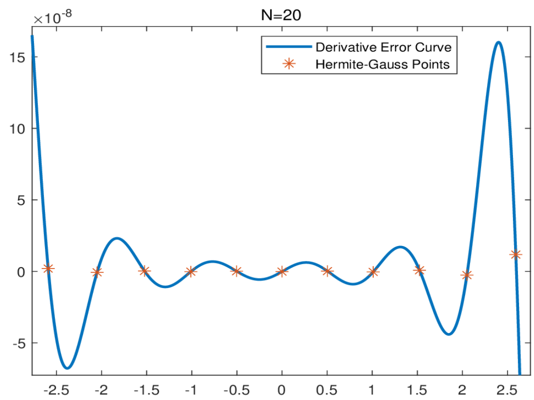

Figure 11: Derivative error: Hermite polynomials, Example 5.1.

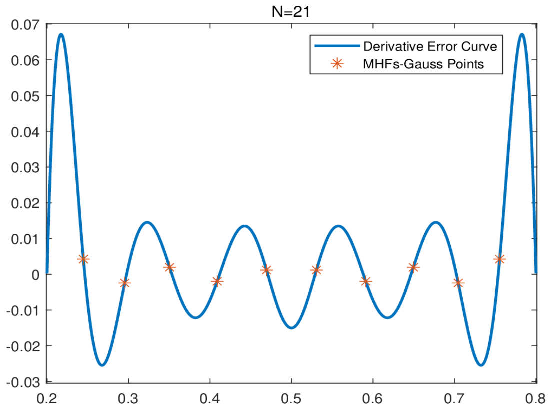

Figure 12: Derivative error: MHFs, Example 5.2.

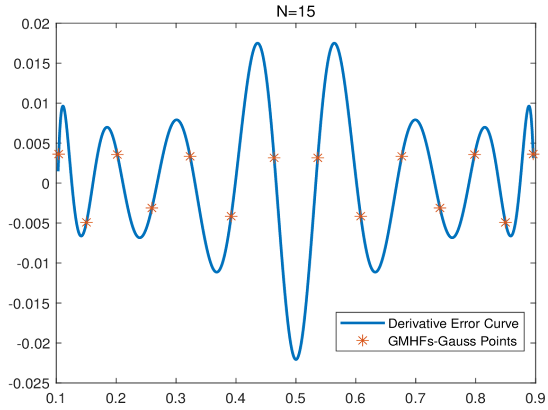

Figure 13: Derivative error: GMHFs, Example 5.2.

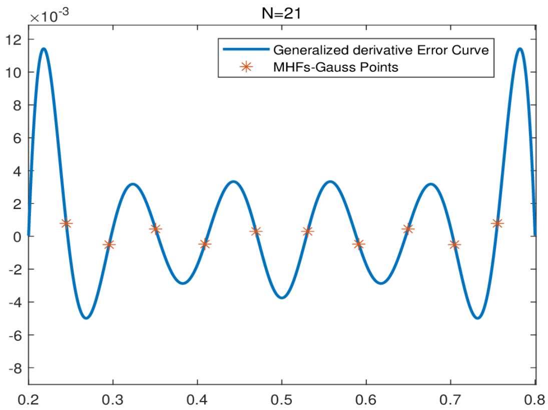

Figure 14: Pseudo derivative error: MHFs, Example 5.2.

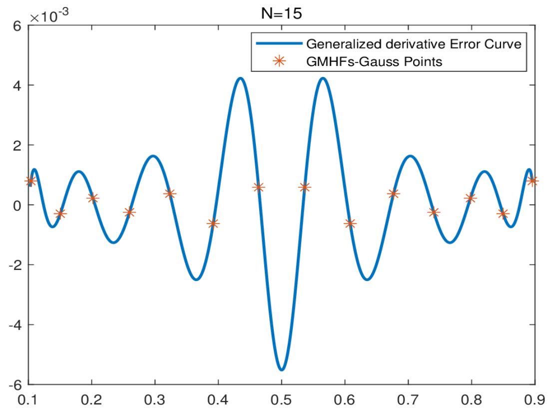

Figure 15: Pseudo derivative error: GMHFs, Example 5.2.

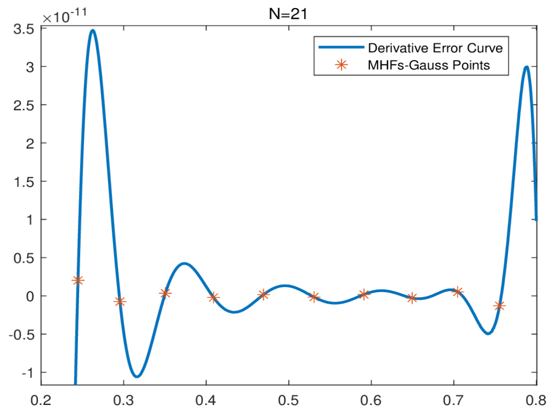

Figure 16: Derivative error: MHFs, Example 5.3.

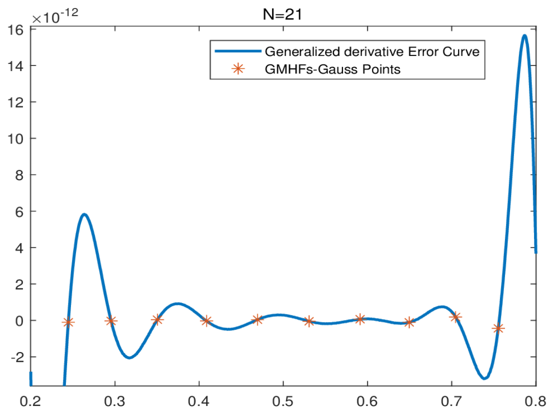

Figure 17: Derivative error: GMHFs, Example 5.3.

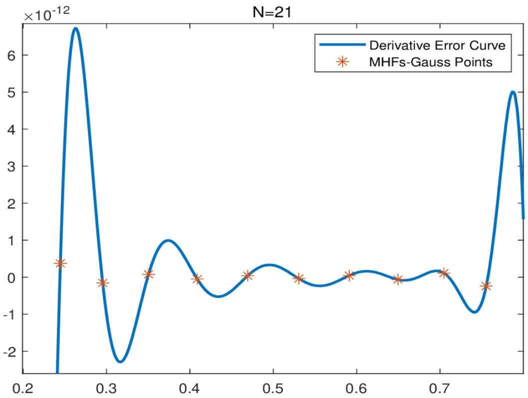

Figure 18: Pseudo derivative error: MHFs, Example 5.3.

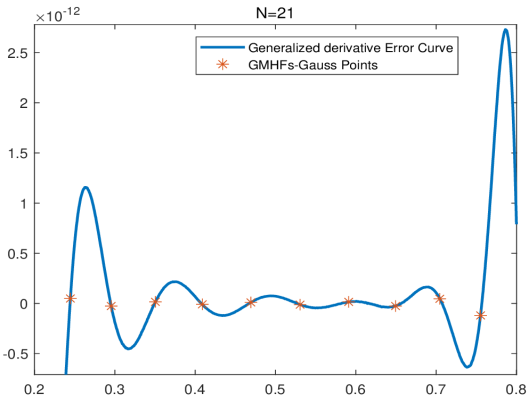

Figure 19: Pseudo derivative error: GMHFs, Example 5.3.

tions and their mapped nonpolynomial functions. Utilizing the contour integral representation of the remainder for Hermite spectral interpolations as described in [32], we examine the superconvergence points of the derivatives of the Hermite spectral interpolations based on the Hermite polynomials/functions. To address multi-point weak singularity problems, we proposed a novel class of mapped Hermite functions (MHFs) which are constructed by applying a mapping to Hermite polynomials in [37]. Since MHFs exhibit rapid growth near $t=0$ and $t=1$, we further introduce generalized mapped Hermite functions (GMHFs) which are much better behaved in this paper. Subsequently, we analyze the superconvergence points of the derivatives of the interpolation errors associated with both MHFs and GMHFs. The results presented in this paper can be applied to solving differential equations or integral equations using Hermite polynomial/function based, mapped Hermite function based, and generalized mapped function based spectral or spectral collocation methods. The data output at the superconvergence points described here could be much more accurate than those at other points.

Appendix A. Some properties of mapped Hermite functions

The properties of mapped Hermite functions:

Proposition A.1 ([37]). *The MHFs have the following properties:*

(1) Three-term recurrence relation

$$\begin{cases} \mathcal{Q}_{n+1}^{(\alpha)}(x)=2\alpha\log\left(\dfrac{x}{1-x}\right)\mathcal{Q}_{n}^{(\alpha)}(x)-2n\mathcal{Q}_{n-1}^{(\alpha)}(x), \quad n\geq 1, \\ \mathcal{Q}_{0}^{((\alpha))}(x)=1, \\ \mathcal{Q}_{1}^{(\alpha)}(x)=2\alpha\log\left(\dfrac{x}{1-x}\right). \end{cases} \tag{A.1}$$

(2) Derivative relations

$$\partial_x \mathcal{Q}_n^{(\alpha)}(x) = \lambda_n \frac{\alpha}{x(1-x)} \mathcal{Q}_{n-1}^{(\alpha)}(x), \tag{A.2}$$

$$\partial_x \mathcal{Q}_n^{(\alpha)}(x) = \frac{2\alpha^2}{x(1-x)} \log\left(\frac{x}{1-x}\right) \mathcal{Q}_n^{(\alpha)}(x) - \frac{\alpha}{x(1-x)} \mathcal{Q}_{n+1}^{(\alpha)}(x), \tag{A.3}$$

where λ_n grows linearly with respect to n, to be precise, $\lambda_n = 2n$.

(3) Orthogonality

$$\int_0^1 \mathcal{Q}_n^{(\alpha)}(x) \mathcal{Q}_m^{(\alpha)}(x) \chi^\alpha(x) dx = \gamma_n^{(\alpha)} \delta_{mn} \tag{A.4}$$

with

$$\chi^\alpha(x) = \frac{e^{-\alpha^2 \log^2(x/(1-x))}}{x(1-x)}, \quad \gamma_n^{(\alpha)} = \frac{\sqrt{\pi} 2^n n!}{\alpha}. \tag{A.5}$$

(4) Sturm-Liouville problem

$$\begin{aligned} &\frac{1}{\alpha^2} x(1-x) \partial_x \left(x(1-x) \partial_x \mathcal{Q}_n^{(\alpha)}(x) \right) \\ &\quad - 2x(1-x) \log\left(\frac{x}{1-x}\right) \mathcal{Q}_n^{(\alpha)}(x) + \lambda_n \mathcal{Q}_n^{(\alpha)}(x) = 0. \end{aligned} \tag{A.6}$$

(5) MHFs-Gauss quadrature.

Let $\{z_j, \omega_j\}_{j=0}^N$ be the Gauss nodes and weights of Hermite polynomial $H_{N+1}(z)$. Denote

$$x_j^{(\alpha)} = \frac{e^{z_j/\alpha}}{1+e^{z_j/\alpha}}, \quad \chi_j^\alpha = \frac{1}{\alpha} \omega_j. \tag{A.7}$$

Then,

$$\int_0^1 f(x) \chi^\alpha(x) dx = \sum_{j=0}^N f\left(x_j^{(\alpha)}\right) \cdot \chi_j^\alpha, \quad \forall f \in \mathcal{P}_{2N+1}^{\log}, \tag{A.8}$$

where

$$\mathcal{P}_k^{\log} := \operatorname{span}\left\{ 1, \log\left(\frac{x}{1-x}\right), \log^2\left(\frac{x}{1-x}\right), \ldots, \log^k\left(\frac{x}{1-x}\right) \right\}.$$

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