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A New Multiphysics Finite Element Method for a Biot Model with Secondary Consolidation

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Abstract. In this paper, we propose a new multiphysics finite element method for a Biot model with secondary consolidation in soil dynamics. To better describe the multiphysical processes underlying in the original model and propose stable numerical methods to overcome "locking phenomenon" of pressure and displacement, we reformulate the swelling clay model with secondary consolidation by a new multiphysics approach, which transforms the fluid-solid coupling problem to a fluid coupled problem. Then, we give the energy law and prior error estimate of the weak solution. Also, we design a fully discrete time-stepping scheme to use multiphysics finite element method with $P_2-P_1-P_1$ element pairs for the space variables and backward Euler method for the time variable, and we derive the discrete energy laws and the optimal convergence order error estimates. Also, we show some numerical examples to verify the theoretical results and there is no "locking phenomenon". Finally, we draw conclusions to summarize the main results of this paper.

AMS subject classifications: 65N30, 65N12

Key words: Biot model, multiphysics finite element method, optimal convergence order, secondary consolidation.

1 Introduction

Biot model with secondary consolidation in soil dynamics plays a very important role in the construction of civil engineering, such as industrial and civil buildings, roads and bridges, water conservancy facilities, embankments and ports (cf. [2, 3, 18, 21, 30]). Compression deformation of saturated clays is usually based on Terzaghi's consolidation theory and Biot's consolidation(cf. [2, 26, 30]), which are the primary consolidation theories.

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However, secondary consolidation is a process in which the volume of saturated clay decreases with time after the completion of primary consolidation, which plays an important role in the study of clay. The control equations of Biot model with secondary consolidation are the same as ones of general poroelasticity model. And the general poroelasticity model is widely applied in various fields such as geophysics, biomechanics, chemical engineering, materials science and so on, one can refer to [2,7,9,13,15,17,22,23,31]. In this paper, we consider the following Biot model with secondary consolidation (quasi-static poroelasticity model, cf. [28]):

$$-\lambda^*\nabla(\mathrm{div}\boldsymbol{\tau})_t-\mathrm{div}\sigma(\boldsymbol{\tau})+b_0\nabla p=\mathbf{F} \quad \text{in } \Omega_T:=\Omega\times(0,T)\subset\mathbb{R}^d\times(0,T), \tag{1.1}$$

$$(a_0p+b_0\mathrm{div}\boldsymbol{\tau})_t+\mathrm{div}\boldsymbol{\zeta}_f=\phi \quad \text{in } \Omega_T, \tag{1.2}$$

where

$$\sigma(\boldsymbol{\tau})=\gamma\varepsilon(\boldsymbol{\tau})+\beta tr\big(\varepsilon(\boldsymbol{\tau})\big)\mathbf{I}, \quad \varepsilon(\boldsymbol{\tau})=\frac{1}{2}\big(\nabla\boldsymbol{\tau}+(\nabla\boldsymbol{\tau})'\big), \tag{1.3}$$

$$\boldsymbol{\zeta}_f:=-\frac{K}{\theta_f}(\nabla p-\rho_f\mathbf{g}). \tag{1.4}$$

Here $\Omega\subset\mathbb{R}^d\,(d=1,2,3)$ denotes a bounded polygonal domain with the boundary $\partial\Omega$. $\boldsymbol{\tau}$ denotes the displacement vector of the solid, p denotes the pressure of the solvent, $\mathbf{I}$ denotes the $d\times d$ identity matrix, $\mathbf{F}$ is the body force. The permeability tensor $K=K(x)$ is assumed to be symmetric and uniformly positive definite in the sense that there exists positive constants K_1 and K_2 such that $K_1|\zeta|^2\leq K(x)\zeta\cdot\zeta\leq K_2|\zeta|^2$ for a.e. $x\in\Omega$ and $\zeta\in\mathbb{R}^d$. The fluid viscosity θ_f, Biot-Willis constant b_0, secondary consolidation coefficient λ^* and the constrained specific storage coefficient a_0 are all non-negative. In addition, $\tilde{\sigma}(\boldsymbol{\tau})$ is called the (effective) stress tensor. $\boldsymbol{\zeta}_f$ is the volumetric fluid flux, which is called Darcy's law. β and γ are Lamé constants, $\widehat{\sigma}(\boldsymbol{\tau},p):=\sigma(\boldsymbol{\tau})-b_0p\mathbf{I}$ is the total stress tensor. We assume that $\rho_f\not\equiv 0$, which is a realistic assumption.

As for Biot model with primary consolidation, Phillips and Wheeler [24] propose and analyze a continuous-in-time linear poroelasticity model. Feng *et al.* [11] propose a multiphysics approach to reformulate the linear poroelasticity model to propose a stable finite element method. The second consolidation is introduced and developed by Cushman and Murad [21]. Showalter [28] finds that the term of $\lambda^*\nabla(\mathrm{div}\boldsymbol{\tau})_t$ has an effect for the momentum equation (1.1) when $\lambda^*>0$ just as the effect for the diffusion equation (1.2) when $a_0>0$. Gaspar [14] introduces a stabilized method for a Biot model with secondary consolidation by using the finite difference method on staggered grids. Lewis and Schrefler [19] use the finite element method to study the Biot model with secondary consolidation but not overcome the "locking phenomenon". In this paper, following the idea of [11], we reformulate the Biot model with secondary consolidation into a fluid coupled problem. A new breakthrough has been made by introducing new variables which are very different from ones of [11] (see Remark 2.1), that is, by introducing the auxiliary variable $q=\mathrm{div}\boldsymbol{\tau}$ and new variables $\varpi=a_0p+b_0q,\delta=b_0p-\beta q-\lambda^*q_t$, we reformulate