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Solving the Inverse Potential Problem in the Parabolic Equation by the Deep Neural Networks Method

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Abstract. In this work, we consider an inverse potential problem in the parabolic equation, where the unknown potential is a space-dependent function and the used measurement is the final time data. The unknown potential in this inverse problem is parameterized by deep neural networks (DNNs) for the reconstruction scheme. First, the uniqueness of the inverse problem is proved under some regularities assumption on the input sources. Then we propose a new loss function with regularization terms depending on the derivatives of the residuals for partial differential equations (PDEs) and the measurements. These extra terms effectively induce higher regularity in solutions so that the ill-posedness of the inverse problem can be handled. Moreover, we establish the corresponding generalization error estimates rigorously. Our proofs exploit the conditional stability of the classical linear inverse source problems, and the mollification on the noisy measurement data which is set to reduce the perturbation errors. Finally, the numerical algorithm and some numerical results are provided.

AMS subject classifications: 34K28, 35R30, 65N15, 62M45

Key words: Inverse potential problem, deep neural networks, uniqueness, generalization error estimates, numerical reconstruction.

1 Introduction

1.1 Mathematical model

The following parabolic system is considered in this work:

$$\begin{cases} \big(\partial_t-\Delta+q(x)\big)u=F(x,t), & (x,t)\in\Omega_T, \\ u(x,t)=b(x,t), & (x,t)\in\partial\Omega_T, \\ u(x,0)=u_0(x), & x\in\Omega. \end{cases} \tag{1.1}$$

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Here we write $\Omega_T=\Omega\times(0,T]$ and $\partial\Omega_T=\partial\Omega\times(0,T]$ for short, and $\Omega\subset\mathbb{R}^d$ is an open bounded domain in $\mathbb{R}^d$ with sufficiently smooth boundary. $F(x,t),u_0(x),b(x,t)$ are the source term, initial status, boundary condition respectively, causing the heat propagation in the medium. The potential function $q(x)\in L^\infty(\Omega)$, called the heat radiative coefficient of the material, is a crucial parameter for characterizing the heat conduction process. It describes the ability of the medium to propagate heat from internal sources or sinks. For known $(F(x,t),u_0(x),b(x,t),q(x))$ with suitable regularities, the forward problem (1.1) is well-posed in appropriate function space [13]. In this work, we consider the inverse problem of recovering the unknown $q(x)$, where the used measurement is the final time data

$$u(x,T):=\varphi(x),\quad x\in\Omega. \tag{1.2}$$

In practical applications of inverse problems, the contamination on inverse problems is unavoidable. So we will be given the noisy data φ^δ instead of the exact data $\varphi(x)$ in (1.2), which satisfies

$$\|\varphi^\delta-\varphi\|_{L^\infty(\Omega)}\le\delta. \tag{1.3}$$

To handle the effect caused by the perturbations, people need to develop effective methods to improve the accuracy and robustness in applications. In this study, we choose the deep neural networks to solve the inverse problem (1.1)-(1.3). Comparing to traditional methods for solving inverse potential problem, this approach demonstrates the superiority in high-dimensional space and has the advantage of breaking the curse of dimensionality.

There are rare works on studying the inverse potential problem for parabolic equations using deep neural networks, especially the rigorous analysis of its convergence estimate. In this work, the authors will consider the solution of the inverse potential problem (1.1)-(1.3) parameterized by DNNs for the reconstruction scheme. We propose a new loss function with regularization terms depending on the derivatives of the residuals for PDEs and measurements. The mollification method has been employed to improve the regularity of the noisy data. Also, the generalization error estimates are rigorously derived from the conditional stability of the linear inverse source problem and the mollification error estimate on noisy data.

1.2 Literature

The reconstructions of $q(x)$ in (1.1) from some inversion input data have been studied extensively. For zero initial status, the uniqueness for $q(x)$ by (1.1)-(1.2) is established in [26], while the unique reconstruction using final measurement data is studied in [27]. In the case of non-zero initial status, the existence and uniqueness of the generalized solution $(u(x,t),q(x))\in W_p^{2,1}(\Omega_T)\times L^p(\Omega)$ with the time-average temperature measurement are given in [20] for (u_0,φ) with some regularities. Choulli and Yamamoto [7] prove the generic well-posedness of the inverse problem in Hölder spaces by final measurement data, and then the conditional stability result in a Hilbert space setting for sufficiently