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Revisiting Hopf Bifurcation in Single-Species Models with Time Delays

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Abstract. Hopf bifurcation in delay differential equations has been a central topic in the study of complex dynamical behaviors in biological and ecological systems. In this review, we revisit Hopf bifurcation phenomena in single-species models that incorporate time delays, emphasizing recent progress in both ordinary and partial differential equation frameworks. We present a comprehensive overview of classic and contemporary models, such as Wright's equation, Nicholson's blowflies equation, and diffusive logistic models, highlighting criteria for local and global bifurcations, the geometric and analytical methods used to determine critical values, and the stability of emerging periodic solutions. The review also covers structured models with age, stage, advection, and spatial effects, as well as equations with multiple delays. Through this survey, we aim to consolidate theoretical insights and provide a unified understanding of delay-induced oscillations in population models, laying the groundwork for future developments in delay-driven dynamics.

AMS subject classifications: 34K18, 35K57, 37N25, 92D25

Key words: Hopf bifurcation, delay differential equations, single-species model, geometric criterion, periodic solutions.

1 Introduction

Time delays are inherent in many natural processes, especially in biological and ecological systems where maturation, gestation, or spatial dispersal are not instantaneous. In population dynamics, even simple single-species models can display rich and complex behavior when delays are incorporated. One particularly important phenomenon is the Hopf bifurcation, where a stable equilibrium loses its stability and gives rise to periodic oscillations as the delay passes a critical threshold [28,70]. The relationship between de-

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lays and complex dynamical behavior has been extensively studied. Ruan [55] offers a thorough analysis of delay differential equations in single-species dynamics, exploring classic models such as Hutchinson's delayed logistic equation and Nicholson's blowflies model, along with their stability and relevance to experimental data. Additionally, the study addresses various factors, including recruitment models, the Allee effect, and multiple delays, emphasizing the intricate dynamics introduced by time delays in biological systems.

This review focuses on Hopf bifurcation in single-species models with various types of delays. We aim to provide an integrated and comprehensive summary of both classical models and recent advances. The review begins with an analysis of the zero distributions of transcendental characteristic equations [58], which form the foundation for identifying bifurcation values. A geometric criterion for determining Hopf bifurcation values is presented, particularly in systems with multiple delays [62].

We then explore several ordinary differential equations with delays that have served as canonical models in the literature. Wright's equation [73], arising from a delayed logistic growth model, is examined in detail, including the historical development and eventual resolution of Wright's and Jones' conjectures [31, 69]. Similarly, Nicholson's blowflies model [25, 51] is discussed along with its extensions that account for maturation delays, mortality during development, and unimodal nonlinear feedback [60, 70]. These examples highlight how increasing biological realism through delay mechanisms can dramatically influence system dynamics, leading to multistability, oscillations, and even chaos.

The review also delves into partial differential equations (PDEs) with delay, where diffusion and spatial heterogeneity interact with delay effects to produce complex dynamics. We analyze models such as the diffusive logistic equation with delay [10, 66, 67], age-structured systems [63], and equations involving advection and habitat gradients [11]. These PDE models often lead to nontrivial spatial patterns and localized oscillations that are biologically relevant.

Throughout the paper, we emphasize analytical techniques such as center manifold theory, Lyapunov-Schmidt reduction, and global Hopf bifurcation theory [19,24,75]. Our goal is to provide a clear synthesis of known results while highlighting areas where further mathematical and computational developments are needed. By revisiting both foundational and contemporary work on Hopf bifurcations in delay systems, we hope to guide researchers toward new challenges and insights in the modeling of real-world biological dynamics.

The remainder of the paper is structured as follows. Section 2 introduces the theoretical background and zero distribution analysis for characteristic equations of delay differential systems. Section 3 reviews ordinary differential equations with delays, highlighting key models such as Wright's equation and Nicholson's blowflies equation. Section 4 shifts focus to partial differential equations with delays, including diffusive logistic models and age-structured systems. In each case, we discuss both local and global Hopf bifurcation results and the nature of the bifurcating solutions.