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Evolutionary Stability in Stochastic Environments

Cong Li[1], Hui Zhang[2], Yi Tao[3,*] and Xiu-Deng Zheng[3,*]

[1] *School of Ecology and Environment, Northwestern Polytechnical University, Xi'an 710072, China.*
[2] *School of Mathematics and Statistics, Northwestern Polytechnical University, Xi'an 710072, China.*
[3] *State Key Laboratory of Animal Biodiversity Conservation and Integrated Pest Management, Institute of Zoology, Chinese Academy of Sciences, Beijing 100101, China.*

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Abstract. Evolutionary stability (or evolutionarily stable strategy) is the core concept of evolutionary game theory. Although evolutionary game theory has achieved great success in evolutionary biology, economics and social science, etc. over the past forty years, how to understand the impact of environmental stochastic fluctuations on the evolutionary stability is still one of the most challenging issues in evolutionary game theory. This paper mainly introduces the recent advances in the study of stochastic evolutionary stability and stochastic Nash equilibrium within the framework of stochastic evolutionary game dynamics. These studies may provide a possible way to better understand the complexity of evolutionary game dynamics in real world situations.

AMS subject classifications: 91A22, 91A15

Key words: Stochastic environments, random payoffs, stochastic evolutionary stability, stochastic Nash equilibrium.

1 Introduction

Forty-three years ago, the publication of J. Maynard Smith's monograph "Evolution and Game Theory" [25] marked the birth of evolutionary game theory as a new fundamental theoretical framework for revealing the evolutionary mechanism of animal behavior. Since then, evolutionary game theory has not only achieved great success in evolutionary biology, but has also been widely applied to economics and social sciences.

The concept of evolutionarily stable strategy (ESS) is the most core theoretical concept in evolutionary game theory, which was introduced by Smith and Price [26] and has

*Corresponding author. *Email addresses:* `yitao@ioz.ac.cn` (Y. Tao), `zhengxd@ioz.ac.cn` (X.-D. Zheng)

become one of the principle tools for analyzing the dynamics of natural selection. An ESS is a strategy such that, if all the members of a population adopt it, then no mutant strategy could invade the population [10,24–26]. In the context of symmetric pairwise interactions occurring random in an infinite population, let $E(\mathbf{x},\mathbf{y})$ represent the payoff received by an individual using strategy $\mathbf{x}$ against an individual using strategy $\mathbf{y}$. Then, $\mathbf{x}$ is an ESS if: (i) $E(\mathbf{x},\mathbf{x}) \geq E(\mathbf{y},\mathbf{x})$ for any strategy $\mathbf{y} \neq \mathbf{x}$, and (ii) $E(\mathbf{x},\mathbf{y}) > E(\mathbf{y},\mathbf{y})$ in the case of an equality in (i). These conditions are necessary and sufficient for the expected payoff to $\mathbf{x}$ to exceed the expected payoff to $\mathbf{y}$ in an infinite population of individuals using $\mathbf{x}$ or $\mathbf{y}$ provided that the frequency of $\mathbf{y}$ is small enough.

In evolutionary game dynamics, the change of the frequency of a strategy in a population over time depends on the difference between the expected payoff of this strategy and the average payoff of the population, that is, dynamics of the strategies frequencies is described by the replicator equation [10, 28]. For n possible strategies, we have

$$\dot{x}_i = x_i\big((\mathbf{A}\mathbf{x})_i - \mathbf{x}\cdot\mathbf{A}\mathbf{x}\big),$$

where $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ is the strategy frequency vector, with x_i being the frequency of strategy i for $i = 1,2,\ldots,n$, and $\mathbf{A} = (a_{ij})_{n\times n}$ is the payoff matrix with a_{ij} being the payoff to strategy i against strategy j for $i,j = 1,2,\ldots,n$. Here it is understood that

$$\mathbf{x}\cdot\mathbf{A}\mathbf{x} = \sum_{i=1}^{n} x_i(\mathbf{A}\mathbf{x})_i = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j a_{ij}$$

with $(\mathbf{A}\mathbf{x})_i$ being the expected payoff to strategy i for $i = 1,2,\ldots,n$. Moreover, if $\mathbf{x}$ is an ESS with respect to the mixed strategies of the n pure strategies with the bilinear payoff function $E(\mathbf{x},\mathbf{y}) = \mathbf{x}\cdot\mathbf{A}\mathbf{y}$, then it is an asymptotically stable rest point of the above replicator dynamics [10].

As it is well known, a Nash equilibrium (NE) is the core concept of non-cooperative games [19]. For linear evolutionary games based on payoff matrix [10, 20, 25], the equilibrium condition for an ESS is exactly the definition of a NE, which is a strategy that is the best reply to itself. In this framework, it is also known that, while an ESS must be a NE, the inverse is not necessarily true. This is the case, however, for a strict NE, which is strictly better against itself than any other strategy. Moreover, if a completely mixed strategy is a NE, then it must be unique and can never be a strict NE [10]. In particular, this implies that it is not possible for two or more completely mixed strategies to be ESSs.

For the situation with finite population size N and with two strategies, denoted by strategy 1 and strategy 2, respectively, Nowak *et al.* [21] proposed to call strategy 1 an ESS_N if two conditions hold when the initial frequency of strategy 2 is N^{-1}: (i) strategy 2 has a lower expected payoff than strategy 1, in which case selection is said to oppose strategy 2 invading strategy 1; and (ii) the probability of ultimate fixation of strategy 2 is less than N^{-1}, in which case selection is said to oppose strategy 2 replacing strategy 1. In general, these conditions depend on the population size N and the reproduction scheme [20].

One key assumption in classical evolutionary game theory is that the payoff matrix is constant, and this supports that the environmental conditions do not change over time. However, this assumption cannot be thought to be always true since environmental conditions in the real world are changing and uncertain. For the population dynamics in ecology, May [18] pointed out that since real environments are uncertain and stochastic, the birth rates, carrying capacities, competition coefficients, and other parameters which characterize natural biological systems all, to a greater or lesser degree, exhibit random fluctuations. One of May's studies reveals profoundly how the stochastic fluctuations of carrying capacity (i.e. environmental stochasticity) influences the dynamics of a single-species population. In fact, effects of environmental stochasticity on population and community ecology have been investigated by many authors [14]. Spagnolo *et al.* [27], for instance, investigated some phenomena in Lokta-Volterra systems induced by environmental noise, for examples, quasi-deterministic oscillations, stochastic resonance, noise-delayed extinction, and spatial pattern.

We noted that the previous work on stochastic evolutionary game theory in an infinite population includes the work of Foster and Young [7], who consider small perturbations to the deterministic replicator dynamics that result from mutations and ordinary chance events affecting the reproductive success of strategies. Then, the dynamics of strategy frequencies obey the stochastic differential equation

$$\dot{x}_i = x_i[(\mathbf{A}\mathbf{x})_i - \mathbf{x}\cdot\mathbf{A}\mathbf{x}] + \sigma[\Gamma(\mathbf{x})\dot{\mathbf{W}}]_i.$$

Here $\dot{\mathbf{W}}$ is a formal time derivative of a standard n-dimensional Brownian motion $\mathbf{W}$, called a white noise, $\Gamma(\mathbf{x})$ is a variance-covariance matrix with all bounded entries and ones on the main diagonal such that $\mathbf{x}\Gamma(\mathbf{x}) = \mathbf{0}$, while $\sigma > 0$ is a parameter representing the strength of the perturbation. For this stochastic dynamical system, a set of States S is called a stochastically stable set (SSS) if, in the long run, it is nearly certain that the system lies within every open set containing S as σ tends to zero. The stochastically stable set is always nonempty and minimizes a suitably defined potential function. However, it is by no means equivalent to the set of evolutionary stable strategies even when the latter exits. It contains often only a subset of the evolutionarily stable strategies, and sometimes even none. So a natural and challenging question is what happens to evolutionary game concepts and dynamics under the effects of a stochastically varying environments.

Another noteworthy work that takes fluctuating environments into account in evolutionary games is the study by Assaf *et al.* [1] in 2013. They posited that fluctuating environments, modeled as, extrinsic noise, stems from the intensity of natural selection. Therefore, they considered the possibility that extrinsic noise could enhance the fixation probability of cooperative behavior in finite population by assuming the intensity of natural selection to be a random variable. They focus on the fixation probability rather than on the dynamics of the system states.

Since environmental conditions in the real world are changing and uncertain, stochastic fluctuations in the surrounding environment of a population may cause changes in the occurrence of interactions between individuals and, more importantly, changes in the payoffs received by the interacting individuals. Therefore, unless stochastic fluctuations are so small that their effects can be neglected, there is no priori reason to assume that the payoff matrix of an evolutionary game is constant if the environment is actually stochastic.

In 2017, in order to explore the impact of environmental stochastic fluctuations on evolutionary game dynamics, Zheng *et al.* [31] (see also [5,17,32]) developed the concept of stochastic evolutionary stability based on conditions for local stochastic stability of stochastic recurrence equations (see also the local stochastic stability of stochastic replicator dynamics [3,4,6,15]). A stochastically evolutionarily stable (SES) strategy is defined as a strategy such that, if all the members of the population adopt it, then the probability for at least any slightly perturbed strategy to successfully invade the population under the influence of natural selection is arbitrarily low. In 2025, based on the analysis of the SES strategy, Li *et al.* [16] introduced the concept of a stochastic Nash equilibrium (SNE) that extends the classical concept of NE, and they address the question of the existence of a SNE, either weak when the geometric mean payoff against it is the same for all other strategies or strong when it is strictly smaller for all other strategies, and its relationship with a SES strategy.

The more detailed descriptions for the stochastic evolutionary stability and for the noise-induced SNE are given in the following, respectively.

2 Stochastic evolutionary stability

For simplicity, consider an evolutionary game in an infinite population with discrete non-overlapping generations. There are two phenotypes or pure strategies, denoted by R_1 and R_2, respectively, and the payoffs in pairwise interactions are given by the payoff matrix

$$\mathbf{A}(t)=\begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix}, \tag{2.1}$$

where $a_{ij}(t)$ is the payoff to strategy R_i against strategy R_j for $i,j=1,2$. These payoffs are assumed to be positive random variables that are uniformly bounded below and above by some positive constants. Therefore, there exist real numbers $A,B>0$ such that $A\le a_{ij}(t)\le B$ for $i,j=1,2$ and for all $t\ge 0$. Moreover, the probability distribution of $a_{ij}(t)$ for $i,j=1,2$ do not depend on $t\ge 0$. The means, variances and covariances of the payoffs are

$$\langle a_{ij}(t)\rangle=\bar{a}_{ij}, \quad \big\langle \big(a_{ij}(t)\big)^2\big\rangle=\sigma_{ij}^2,$$

and

$$\big\langle \big(a_{ij}(t)-\bar{a}_{ij}\big)\big(a_{kl}(t)-\bar{a}_{kl}\big)\big\rangle=\sigma_{ij,kl},$$

respectively, for $i,j,k,l=1,2$ with $(i,j)\neq(k,l)$, where the symbol $\langle\cdot\rangle$ denotes the mathematical expectation. As for $s\neq t$, the payoffs $a_{ij}(s)$ and $a_{kl}(t)$ are assumed to be independent so that

$$\big\langle \big(a_{ij}(s)-\bar{a}_{ij}\big)\big(a_{kl}(t)-\bar{a}_{kl}\big)\big\rangle=0, \quad i,j,k,l=1,2.$$

Consider a population consisting of individuals using only two mixed strategies $\mathbf{x}=(x,1-x)$ and $\hat{\mathbf{x}}=(\hat{x},1-\hat{x})$ with $x,\hat{x}\in[0,1]$. The payoff matrix for these two mixed strategies at time step $t\geq 1$ is given by

$$\begin{pmatrix} \mathbf{x}\cdot\mathbf{A}(t)\mathbf{x} & \mathbf{x}\cdot\mathbf{A}(t)\hat{\mathbf{x}} \\ \hat{\mathbf{x}}\cdot\mathbf{A}(t)\mathbf{x} & \hat{\mathbf{x}}\cdot\mathbf{A}(t)\hat{\mathbf{x}} \end{pmatrix}, \tag{2.2}$$

where the entry $\mathbf{x}\cdot\mathbf{A}(t)\hat{\mathbf{x}}$ is the payoff to strategy $\mathbf{x}$ against strategy $\hat{\mathbf{x}}$, and similarly for the other entries.

Let $q(t)$ be the frequency of strategy $\mathbf{x}$ in the population at time step t. Assuming random pairwise interactions, the average payoffs to strategies $\mathbf{x}$ and $\hat{\mathbf{x}}$ at time step t are given by

$$\begin{aligned} \pi_{\mathbf{x}}(t) &= q(t)\mathbf{x}\cdot\mathbf{A}(t)\mathbf{x}+\big(1-q(t)\big)\mathbf{x}\cdot\mathbf{A}(t)\hat{\mathbf{x}}, \\ \pi_{\hat{\mathbf{x}}}(t) &= q(t)\hat{\mathbf{x}}\cdot\mathbf{A}(t)\mathbf{x}+\big(1-q(t)\big)\hat{\mathbf{x}}\cdot\mathbf{A}(t)\hat{\mathbf{x}}. \end{aligned} \tag{2.3}$$

Taking the average payoff as fitness [10], the frequency of strategy $\mathbf{x}$ at time step $t+1$ can be expressed as

$$q(t+1)=\frac{q(t)\pi_{\mathbf{x}}(t)}{q(t)\pi_{\mathbf{x}}(t)+\big(1-q(t)\big)\pi_{\hat{\mathbf{x}}}(t)}, \tag{2.4}$$

which is a stochastic recurrence equation.

By the definition of stochastic evolutionary stability, a SES strategy is defined as a strategy such that, if all members of the population adopt it, then the probability for at least any slightly perturbed strategy to successfully invade the population under the influence of natural selection is arbitrarily low [31], the mixed strategy $\hat{\mathbf{x}}$ is a SES strategy if the boundary $q(t)=0$ of Eq. (2.4) is stochastically locally stable (SLS) for all possible $\mathbf{x}\neq\hat{\mathbf{x}}$ [16, 31]. For the definition of stochastic local stability, a constant equilibrium q^* of Eq. (2.4) is said to be SLS if for any $\epsilon>0$ there exists $\delta>0$ such that $\mathbb{P}(q_t\to q^*)\geq 1-\epsilon$ as soon as $|q_0-q^*|<\delta$ [2, 12, 13]. It means that q_t tends to q^* as $t\to\infty$ with probability arbitrarily close to 1 if the initial state of the system q_0 is sufficiently close to q^*. Therefore, it can be shown that $\hat{\mathbf{x}}$ is SES if and only if

$$\big\langle \log\big(\mathbf{x}\cdot\mathbf{A}(t)\hat{\mathbf{x}}\big)\big\rangle \leq \big\langle \log\big(\hat{\mathbf{x}}\cdot\mathbf{A}(t)\hat{\mathbf{x}}\big)\big\rangle \tag{2.5}$$

for all possible $\mathbf{x}$, and

$$\left\langle \frac{\hat{\mathbf{x}}\cdot\mathbf{A}(t)\mathbf{x}}{\hat{\mathbf{x}}\cdot\mathbf{A}(t)\hat{\mathbf{x}}}\right\rangle - \left\langle \frac{\mathbf{x}\cdot\mathbf{A}(t)\mathbf{x}}{\hat{\mathbf{x}}\cdot\mathbf{A}(t)\hat{\mathbf{x}}}\right\rangle = -(\hat{x}-x)^2 D>0 \tag{2.6}$$

with

$$D=\left\langle \frac{\big(a_{11}(t)-a_{12}(t)-a_{21}(t)+a_{22}(t)\big)^2}{a_{11}(t)a_{22}(t)-a_{12}(t)a_{21}(t)}\right\rangle \tag{2.7}$$

in the case of an equality in Eq. (2.5) for all possible $\mathbf{x}\neq\hat{\mathbf{x}}$. For the mathematical proofs of Eqs. (2.5)-(2.7), please see Zheng *et al.* [31] and Li *et al.* [16].

Furthermore, the stochastic evolutionary stability in the stochastic replicator dynamics is investigated by Feng *et al.* [4].

3 Noise-induced SNE

By analogy with the conditions for equilibrium and stability of an ESS (see [10, p. 63]), the condition in Eq. (2.5) is used to define a SNE. This corresponds to a strategy that is the best reply to itself in a stochastic environment based on the geometric means of the payoffs rather than their arithmetic means.

Let us recall that the geometric mean of a random variable $a>0$ is defined as $GM(a)=\exp(\langle \log a\rangle)$. In terms of geometric means, Eq. (2.5) is equivalent to

$$GM\big(\mathbf{x}\cdot\mathbf{A}(t)\hat{\mathbf{x}}\big)\le GM\big(\hat{\mathbf{x}}\cdot\mathbf{A}(t)\hat{\mathbf{x}}\big) \tag{3.1}$$

for all possible $\mathbf{x}$. This is the condition for the mixed strategy $\hat{\mathbf{x}}$ to be a SNE. Then,

(i) In the special case where

$$GM\big(\mathbf{x}\cdot\mathbf{A}(t)\hat{\mathbf{x}}\big)= GM\big(\hat{\mathbf{x}}\cdot\mathbf{A}(t)\hat{\mathbf{x}}\big) \tag{3.2}$$

for all $\mathbf{x}\neq\hat{\mathbf{x}}$, the strategy $\hat{\mathbf{x}}$ will be called a weak stochastic Nash equilibrium (weak SNE).

(ii) At the other extreme, if

$$GM\big(\mathbf{x}\cdot\mathbf{A}(t)\hat{\mathbf{x}}\big)< GM\big(\hat{\mathbf{x}}\cdot\mathbf{A}(t)\hat{\mathbf{x}}\big) \tag{3.3}$$

for all $\mathbf{x}\neq\hat{\mathbf{x}}$, then $\hat{\mathbf{x}}$ will be said a strong stochastic Nash equilibrium (strong SNE). Note that the SNE condition is necessary but not sufficient for stochastic evolutionary stability, while the condition for a strong SNE is sufficient but not necessary.

In order to distinguish the pure strategies R_1 and R_2, we here assume that

$$\mathbb{P}\big(a_{11}(t)=a_{21}(t),a_{22}(t)=a_{12}(t)\big)<1.$$

For the equilibrium structure of the system, we have that

(i) A mixed strategy $\mathbf{x}^*=(x^*,1-x^*)$ such that

$$\mathbb{P}\big(\big(\mathbf{A}(t)\mathbf{x}^*\big)_1=\big(\mathbf{A}(t)\mathbf{x}^*\big)_2\big)<1$$

is a strong SNE if and only if

$$\left\langle \frac{\big(\mathbf{A}(t)\mathbf{x}^*\big)_1-\big(\mathbf{A}(t)\mathbf{x}^*\big)_2}{\mathbf{x}^*\cdot\mathbf{A}(t)\mathbf{x}^*}\right\rangle \begin{cases}\le 0, & \text{if} \quad x^*=0,\\ \ge 0, & \text{if} \quad x^*=1,\\ =0, & \text{if} \quad 0<x^*<1,\end{cases} \tag{3.4}$$

where we can see that the pure strategy R_1, or R_2, is a strong SNE if and only if

$$\left\langle\frac{a_{21}(t)}{a_{11}(t)}\right\rangle\le 1, \quad \text{or} \quad \left\langle\frac{a_{12}(t)}{a_{22}(t)}\right\rangle\le 1.$$

Moreover, if

$$\mathbb{P}\big((\mathbf{A}(t)\mathbf{x})_1 = (\mathbf{A}(t)\mathbf{x})_2\big) < 1$$

for all possible $\mathbf{x}$, then there exists at least one strong SNE $\mathbf{x}^*$, which is necessarily SES.

(ii) If

$$\mathbb{P}\big((\mathbf{A}(t)\hat{\mathbf{x}})_1 = (\mathbf{A}(t)\hat{\mathbf{x}})_2\big) = 1$$

for some $\hat{\mathbf{x}} = (\hat{x}, 1-\hat{x})$, then $\hat{\mathbf{x}}$ is a weak SNE with an equality in Eq. (2.5) for all possible $\mathbf{x}$, and it is unique weak SNE in the system. For the case with $\mathbb{P}(a_{11}(t) = a_{21}(t)) = 1$, the unique weak SNE is $\hat{\mathbf{x}} = (1,0)$ and it is SES if

$$D = \left\langle \frac{a_{22}(t) - a_{12}(t)}{a_{11}(t)} \right\rangle < 0. \tag{3.5}$$

Analogously, for the case with $\mathbb{P}(a_{22}(t) = a_{12}(t)) = 1$, the unique weak SNE is $\hat{\mathbf{x}} = (0,1)$, which is SES if

$$D = \left\langle \frac{a_{11}(t) - a_{21}(t)}{a_{22}(t)} \right\rangle < 0\ . \tag{3.6}$$

Finally, if there exists $r > 0$ such that

$$\mathbb{P}\big(a_{11}(t) - a_{21}(t) = r\big(a_{22}(t) - a_{12}(t)\big)\big) = 1,$$

then $\hat{\mathbf{x}} = (1/(1+r), r/(1+r))$ is the unique weak SNE, and it is SES if

$$D = \left\langle \frac{a_{22}(t) - a_{12}(t)}{ra_{22}(t) + a_{21}(t)} \right\rangle < 0. \tag{3.7}$$

(iii) In the case where $D > 0$ in the three cases in (ii), there exists at least one strong SNE $\mathbf{x}^* = (x^*, 1-x^*)$ in the first two cases, and even at least two strong SNEs $\mathbf{x}_1^* = (x_1^*, 1-x_1^*)$ and $\mathbf{x}_2^* = (x_2^*, 1-x_2^*)$ in the third case. As for $D = 0$ in the three cases, defining the quantity

$$\hat{u} = \left\langle \frac{\alpha(t)^3 \beta(t)}{\big(\det \mathbf{A}(t)\big)^2} \right\rangle \Big/ \left\langle \frac{\alpha(t)^4}{\big(\det \mathbf{A}(t)\big)^2} \right\rangle, \tag{3.8}$$

where

$$\begin{aligned}
&\alpha(t) = a_{11}(t) - a_{12}(t) - a_{21}(t) + a_{22}(t), \\
&\beta(t) = a_{22}(t) - a_{21}(t),
\end{aligned}$$

it can be shown that there is at least one strong SNE $\mathbf{x}^* = (x^*, 1-x^*)$ with $x^* \in [0, \hat{x})$ if $\hat{x} > \max(0, \hat{u})$, or $x^* \in (\hat{x}, 1]$ if $\hat{x} < \min(1, \hat{u})$.

The above descriptions for the equilibrium structure of the system show that there always exists at least one SNE, weak or strong, and one or two strong SNEs, necessarily

SES, can coexist with a weak SNE. For the proofs of the results in (i)-(iii), please see Li *et al.* [16].

As a simple example (see also Li *et al.* [16]), the random payoff matrix is given by

$$\mathbf{A}(t)=\begin{pmatrix}\mu+a & \mu+a\\ \mu & \mu+\xi(t)\end{pmatrix}. \tag{3.9}$$

Here, a and μ are positive constants with μ small enough but $\mu\neq 0$, while $\xi(t)$ is a non-negative random variable with $\xi(t)=b>a$ with probability p and $\xi(t)=0$ with probability $1-p$, so that

$$\bar{\xi}=\langle\xi(t)\rangle=pb,\quad \sigma_{\xi}^2=\left\langle\left(\xi(t)-\bar{\xi}\right)^2\right\rangle=p(1-p)b^2.$$

Note that

$$\left\langle\frac{\left(\mathbf{A}(t)\mathbf{x}\right)_1-\left(\mathbf{A}(t)\mathbf{x}\right)_2}{\mathbf{x}\cdot\mathbf{A}(t)\mathbf{x}}\right\rangle=\left\langle\frac{a-(1-x)\xi(t)}{\mu+ax+(1-x)^2\xi(t)}\right\rangle \tag{3.10}$$

for all possible $\mathbf{x}=(x,1-x)$. Thus, the strategy $\mathbf{x}^*=(1,0)$ must be a strong SNE since $a/(a+\mu)>0$, while the strategy $\mathbf{x}^*=(0,1)$ is a strong SNE if and only if

$$\frac{a-b}{\mu+b}p+\frac{a}{\mu}(1-p)\leq 0,$$

which cannot occur. As for a strong SNE $\mathbf{x}^*=(x^*,1-x^*)$ with $x^*\in(0,1)$, it must be a solution of the equation

$$\left(\frac{a-(1-x^*)b}{\mu+ax^*+(1-x^*)^2b}\right)p+\left(\frac{a}{\mu a x^*}\right)(1-p)=0. \tag{3.11}$$

Since μ is assumed to be small, the above equation can be approximated as

$$bx^{*2}-\left((2-p)b-a\right)x^*+(1-p)b=0, \tag{3.12}$$

whose solutions are

$$x_{1,2}^*=\frac{b(2-p)-a\pm\sqrt{(bp+a)^2-4ab}}{2b}\in(0,1) \tag{3.13}$$

under the condition that

$$p\geq\frac{2\sqrt{ab}-a}{b}\in(0,1). \tag{3.14}$$

Therefore, for p large enough, there may exist up to two strong SNEs besides the pure strategy R_1, both of which are noise-induced since they do not exist for small p. The simulations results are presented in Fig. 1, which are obtained through repeated simulations based on [16, Fig. 1]. Here, we take $a=4, b=10$ and $\mu=0.01$ for the payoffs in Eq. (3.9). Initially, the population is composed entirely of individuals using strategy $\mathbf{x}=(x,1-x)$. At each time step, a randomly generated mutant strategy $\mathbf{v}=(v,1-v)$ with $v\in[0,1]$ emerges in the population with a probability of 0.01 and accounts for a proportion 0.001. The population evolves in accordance with the stochastic recurrence equation, and a strategy is removed when its proportion falls below 0.0005. The color of each point on the $p-x$ plane represents the average proportion of the initial strategy $\mathbf{x}$ that still retained in the population after 10^4 time steps across 10^2 runs. As can be seen from Fig. 1, when p is sufficiently large (i.e. $p>0.865$, see Eq. (3.14)), there exist three strong SNEs: the black and white dashed curves, both of which are noise-induced, and the red line, which represents the pure strategy $\mathbf{x}=(1,0)$.

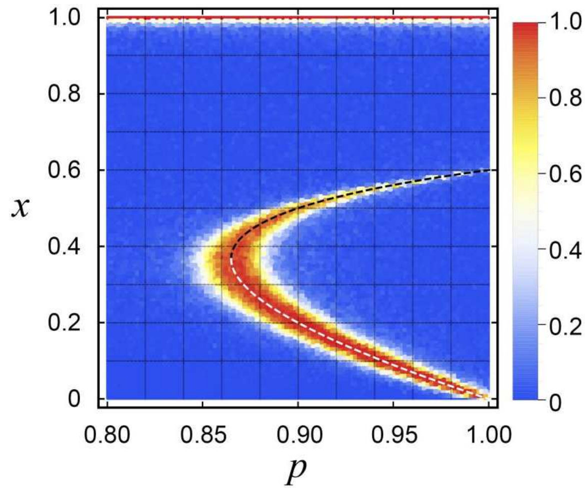

Figure 1: Stochastic simulations for the existence of noise-induced strong SNEs. We take $a=4,b=10$ and $\mu=0.01$ for the payoff matrix given in Eq. (3.9). The horizontal axis denotes the value of p, and the vertical axis the initial strategy $\mathbf{x}=(x,1-x)$ in the population. The color of each point on the p - x plane denotes the average proportion of the initial strategy $\mathbf{x}$ still retained in the population after 10^4 time steps in 10^2 runs. The black, white dashed curves and the pure strategy $\mathbf{x}=(1,0)$ represent the three strong SNEs theoretically predicted.

4 Conclusion

In order to show how to explore the impact of environmental stochastic fluctuations on the evolutionary game dynamics, we focus here on the recent advances for the concepts of the stochastic evolutionary stability (SES strategy) and the stochastic Nash equilibrium in the matrix games, which are defined in a stochastic framework and fully cover the classic concepts of the evolutionary stable strategy and the Nash equilibrium in the deterministic setting. Similar to the relationship between ESS and NE in the deterministic evolutionary game dynamics [10], for the stochastic evolutionary game dynamics, the SNE condition is necessary but not sufficient for stochastic evolutionary stability, while the condition for a strong SNE is sufficient but not necessary, that is, for a linear evolutionary game dynamics with random payoffs, we have:

(i) at least one SNE exists,
(ii) a SES strategy must be a SNE,
(iii) a strong SNE must be a SES strategy, but this is not necessarily the case for a weak SNE,
(iv) a strong SNE can be a completely mixed strategy,
(v) more than one strong SNE can exist.

Furthermore, we can see also that a SNE may have some properties that a NE cannot possess. For instance, in deterministic evolutionary game dynamics, a completely mixed strategy cannot be a strict NE (strong NE in our terminology), while a completely mixed NE must correspond to an interior equilibrium in the evolutionary dynamics of pure strategies [10]. On the contrary, a completely mixed strategy can be a strong SNE for the stochastic evolutionary game dynamics, but it must not correspond to an interior equilibrium in the stochastic evolutionary game dynamics of pure strategies [16].

Ultimately, we posit that the concepts of stochastic evolutionary stability and stochastic Nash equilibrium developed in matrix games with random payoffs may offer a novel and robust means for enhancing our understanding of the complexity of evolutionary game dynamics within real-world scenarios. Furthermore, these two concepts should be potentially extended to other forms of games, such as public goods game, etc. [9, 11, 20, 22], and the games that involve the feedback-coupled systems, called also as ecoevolutionary dynamics [8, 23, 29, 30].

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