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CHOICE OF INTERIOR PENALTY COEFFICIENT FOR INTERIOR PENALTY DISCONTINUOUS GALERKIN METHOD FOR BIOT'S SYSTEM BY EMPLOYING MACHINE LEARNING

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Abstract. This paper uses neural networks and machine learning to study the optimal choice of the interior penalty parameter of the discontinuous Galerkin finite element methods for both the elliptic problems and Biot's systems. It is crucial to choose the optimal interior penalty parameter, which is not too small or too large for the stability, robustness, and efficiency of the approximated numerical solutions. Both linear regression and nonlinear artificial neural network methods are employed and compared using several numerical experiments to illustrate the capability of our proposed computational framework. This framework is integral to developing automated numerical simulation because it can automatically identify the optimal interior penalty parameter. Real-time feedback could also be implemented to update and improve model accuracy on the fly.

Key words. Discontinuous Galerkin, interior penalty, neural networks, machine learning, finite element methods.

1. Introduction

Discontinuous Galerkin (DG) finite element method, which is also known as the Interior Penalty method (IP), is one of the most popular non-conforming finite elements employed for various realistic applications, especially with discontinuous material properties [1, 2, 3, 4, 5]. The advantages of the IP-DG method include the following. First, DG preserves the local flux conservation with highly varying material properties [6, 7, 8, 9, 10]. In addition, DG can deal robustly with general partial differential equations and equations whose type changes within the computational domain, such as from advection dominated to diffusion dominated [11, 12, 13].

However, one of the disadvantages of DG is that the method's stability and accuracy depend on the interior penalty parameter in front of the jump term that needs to be chosen. In other words, the performance of the DG methods highly depends on the choice of the interior penalty parameter. For example, if the penalty parameter is too small, the stability of the scheme is not guaranteed, and the linear solver will not converge. If the penalty parameter is too large, DG schemes might converge to the continuous Galerkin finite element methods and often suffer from the linear solver. Thus, it is crucial to employ the optimal interior penalty parameter. Several studies of the lower bounds for the penalty parameter have been obtained in the past [14, 15, 16, 17, 18]. Moreover, weighted interior penalty parameters for the cases where the diffusion coefficient is discontinuous were studied in [19, 20]. Specific illustrations on the selection of the penalty parameters are shown in [21].

This paper proposes a new procedure to find the optimal interior penalty parameters for both elliptic problems and the poroelastic Biot's system. Since the choice of the optimal interior penalty parameters for multiphysics multiscale coupled problems or problems with discontinuous and heterogeneous material properties are

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nontrivial by the traditional analytic approaches, we employ machine learning processes to predict the optimal interior penalty parameters. Many machine learning models have been a center of attention for decades because of their approximation power that could be practically applied to various applications [22, 23]. These algorithms range from classic linear regression models [24, 25], spatial interpolation techniques such as kriging [26] or maximum likelihood estimation [27], and nonlinear approximation functions such nonlinear regression [28] or deep learning [29].

Recently, deep learning has become more attractive with several advantages, including that it is scalable [30], suitable for GPU functionality [31], and requires less computational resources is less due to the mini-batch gradient descent approach [32]. Deep learning has also been successfully applied to solve partial differential equations, generally solved by classical numerical methods such as finite difference, finite volume, or finite element methods [33, 34, 35, 36, 37, 38]. Moreover, this technique has been used to assist the traditional numerical methods such as finite elements to enhance their performance [39, 40, 41, 42, 43]. Hence, this paper aims to apply this method to identify the optimal interior penalty parameters in complex problems.

The proposed procedure benefits the simple elliptic problem or Biot's equations and any multiphysics multiscale coupled problems. Besides, in cases where many simulations have to be performed with different settings, e.g., mesh size, material properties, or various interior penalty schemes, our proposed framework can automatically identify the optimal interior penalty parameter. Real-time feedback could also be implemented to update and improve model accuracy.

The paper is organized as follows. Our governing system and finite element discretizations are in Section 2 and Section 3, respectively. Details about the machine learning algorithm are discussed in Section 4. The numerical results are in Section 5; this section illustrates the effects of interior penalty parameters on both solution quality and simulation behavior. Performance between linear and nonlinear approximation functions is also compared. Finally, the conclusions follow in Section 6.

2. Mathematical Model

In this section, we briefly recapitulate the Biot system for poroelasticity that we will discuss in this paper. Let $\Omega \subset \mathbb{R}^d$ ($d \in \{1, 2, 3\}$) be the computational domain, which is bounded by the boundary, $\partial\Omega$. The time domain is denoted by $\mathbb{T} = (0, T]$ with $T > 0$. Then the coupling between the fluid flow and solid deformation can be captured by applying Biot's equation of poroelasticity, which is composed of linear momentum and mass balance equations [44].

First, the mass balance equation is given as [45]:

$$\rho\left(\phi c_f + \frac{\alpha - \phi}{K_s}\right)\frac{\partial}{\partial t}p + \rho\alpha\frac{\partial}{\partial t}\nabla\cdot\mathbf{u} - \nabla\cdot\boldsymbol{\kappa}(\nabla p - \rho\mathbf{g}) = g \text{ in } \Omega\times\mathbb{T}, \tag{1}$$

where $p(\cdot, t) : \Omega\times(0; T] \to \mathbb{R}$ is a scalar-valued fluid pressure, $\mathbf{u}(\cdot, t) : \Omega\times(0; T] \to \mathbb{R}^d$ is a vector-valued displacement, ρ is a fluid density, ϕ is an initial porosity, c_f is a fluid compressibility, $\mathbf{g}$ is a gravitational vector, g is a sink/source. Here, $\nabla\cdot\mathbf{u}$ term represents the volumetric deformation and $\boldsymbol{\kappa}$ is defined as:

$$\boldsymbol{\kappa} := \frac{\rho\mathbf{k}_m}{\mu}, \tag{2}$$

where $\mathbf{k}_m$ is a matrix permeability tensor and μ is a fluid viscosity.