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A KRASNOSELSKII-MANN PROXIMITY ALGORITHM FOR MARKOWITZ PORTFOLIOS WITH ADAPTIVE EXPECTED RETURN LEVEL

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Abstract. Markowitz's criterion aims to balance expected return and risk when optimizing the portfolio. The expected return level is usually fixed according to the risk appetite of an investor, then the risk is minimized at this fixed return level. However, the investor may not know which return level is suitable for her/him and the current financial circumstance. It motivates us to find a novel approach that adaptively optimizes this return level and the portfolio at the same time. It not only relieves the trouble of deciding the return level during an investment but also gets more adaptive to the ever-changing financial market than a subjective return level. In order to solve the new model, we propose an exact, convergent, and efficient Krasnoselskii-Mann Proximity Algorithm based on the proximity operator and Krasnoselskii-Mann momentum technique. Extensive experiments show that the proposed method achieves significant improvements over state-of-the-art methods in portfolio optimization. This finding may contribute a new perspective on the relationship between return and risk in portfolio optimization.

Key words. Markowitz portfolio, adaptive expected return, ℓ^1 regularization, Krasnoselskii-Mann algorithm.

1. Introduction

Portfolio optimization (PO) with machine learning methods has become a prospective approach in advancing the interdiscipline of financial engineering [1, 2, 3, 4]. Ever since the first proposal of the mean-variance (MV) approach by Markowitz [5], his criterion has become the most popular one for many PO models [6, 7, 8, 9, 10, 11]. In brief, the original MV (OMV) model is

$$\tag{1} \begin{aligned} \hat{\boldsymbol{w}} &= \arg\min_{\boldsymbol{w}\in\mathbb{R}^N} \boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w}, \\ &\text{s. t. } \boldsymbol{w}^\top \mathbf{1}_N = 1,\ \boldsymbol{w}^\top \boldsymbol{\mu} = \rho, \end{aligned}$$

where $\boldsymbol{w}$ denotes the N-dimensional portfolio (with respect to N assets); $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ denote the expected return and the return covariance of these N assets, respectively. Constraint $\boldsymbol{w}^\top \mathbf{1}_N = 1$ ($\mathbf{1}_N$ denotes the vector of N ones) is the self-financing constraint, which indicates that no additional money can be used and full re-investment is compulsory. Constraint $\boldsymbol{w}^\top \boldsymbol{\mu} = \rho$ means that the expected portfolio return is fixed at a level of ρ. The objective is to minimize the portfolio variance $\boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w}$ (considered as the portfolio risk) at this return level.

Based on many theoretical and practical milestone researches in finance, such as the Capital Asset Pricing Model (CAPM, [12]), the mutual fund performance [13] and the efficient market theory [14], a higher portfolio return $\boldsymbol{w}^\top \boldsymbol{\mu}$ accompanies a higher portfolio risk $\boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w}$. Thus they are usually treated as a pair, and the corresponding Pareto optimals form the efficient frontier [12] of all the feasible portfolios. In this sense, different individuals may choose different return levels ρ

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according to their risk appetites. This convention continues in both theoretical and practical portfolio management today.

On the other hand, machine learning methods have been extending the methodology scope of PO. For example, sparsity methods have been employed to increase portfolio concentration. Brodie et al. [9] impose ℓ^1-regularization [15, 16] on the portfolio to make it sparse and stable. Lai et al. [2] adopt the alternating direction method of multipliers (ADMM, [17]) to solve a short-term sparse PO model. Luo et al. [18] find several closed-form solutions for a short-term sparse PO model with ℓ^0-regularization. Different from constructing a sparse portfolio, Lai et al. [3] focus on covariance estimation in PO. They construct a covariance estimate in the perspective of operators and operator spaces. The latter 3 methods are based on the Exponential Growth Rate (EGR) criterion [19, 20], which has a different investing philosophy from the MV criterion [4]. Therefore, machine learning methods for the MV criterion are still in great demand.

In the perspective of machine learning, we are inspired to investigate whether it is possible to use an adaptive and flexible return level ρ that fits the ever-changing financial market. It also makes sense in finance: the investor may have no idea about what return level ρ is suitable for her/him, or for the current financial market; All he/she wants may be just getting a reasonable return from the market and getting rid of the trouble to choose a subjective ρ. Nevertheless, it is nontrivial to optimize ρ and $\boldsymbol{w}$ simultaneously, especially to achieve satisfactory investing performance. It motivates us to develop a novel PO model named Markowitz Portfolio with Adaptive Expected Return Level (MPAERL), which can dynamically balance return and risk. Our main contributions can be summarized as follows.

1) We develop a new PO model with adaptive expected return level, which including ℓ^1-regularization, equality constraints and inequality constraints.
2) We propose a convergent and efficient Krasnoselskii-Mann Proximity Algorithm (KMPA) which based on the proximity operator and the Krasnoselskii-Mann momentum technique to solve this new PO model.
3) Our proposed KMPA can be directly extended to solve a class of two-term convex optimization models with inequality constraints.

The rest of this paper presents the following contents. Section 2 introduces some related works in this field. Section 3 establishes the MPAERL model. In section 4, we develop an efficient Krasnoselskii-Mann Proximity Algorithm (KMPA) to solve the MPAERL model. We analyze the convergence of the KMPA in section 5. Section 6 conducts extensive experimental results to assess the performance of MPAERL. Section 7 draws conclusions. Last, we provide the proofs of some technical results in the appendices.

2. Related Works

Brodie et al. [9] propose the Sparse and Stable Markowitz Portfolios (SSMP) formulated in Lasso [15]

$$\begin{aligned} \hat{\boldsymbol{w}} &= \operatorname*{arg\,min}_{\boldsymbol{w}\in\mathbb{R}^N} \left\{ \frac{1}{T}\|\boldsymbol{R}\boldsymbol{w} - \rho\mathbf{1}_T\|_2^2 + \tau\|\boldsymbol{w}\|_1 \right\}, \\ &\text{s. t. } \boldsymbol{w}^\top\hat{\boldsymbol{\mu}} = \rho, \boldsymbol{w}^\top\mathbf{1}_N = 1, \end{aligned} \tag{2}$$

where $\boldsymbol{R} \in \mathbb{R}^{T\times N}$ is the sample asset return matrix (T trading times and N assets), $\boldsymbol{r}^{(t)}$ denotes the t-th row of $\boldsymbol{R}$ (i.e., the asset returns at time t), $\hat{\boldsymbol{\mu}} := \frac{1}{T}\boldsymbol{R}^\top\mathbf{1}_T$ is a column vector of sample mean returns, $\rho \in \mathbb{R}$ is a given expected return level, $\tau \geqslant 0$ is the regularization parameter, $\|\cdot\|_2$ is the ℓ^2-norm and $\|\cdot\|_1$ is the ℓ^1-norm.