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ON A 1/2-EQUATION MODEL OF TURBULENCE

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Abstract. In 1-equation URANS models of turbulence, the eddy viscosity is given by $\nu_T = 0.55l(x,t)\sqrt{k(x,t)}$. The length scale l must be pre-specified and $k(x,t)$ is determined by solving a nonlinear partial differential equation. We show that in interesting cases the spacial mean of $k(x,t)$ satisfies a simple ordinary differential equation. Using its solution in ν_T results in a 1/2-equation model. This model has attractive analytic properties. Further, in comparative tests in 2d and 3d the velocity statistics produced by the 1/2-equation model are comparable to those of the full 1-equation model.

Key words. Turbulence, eddy viscosity model, and 1-equation model.

1. Introduction

Unsteady Reynolds-averaged Navier-Stokes (URANS) models can be developed in several ways. Here we adopt finite time averaging that does not require assumptions of ergodicity. Other approaches, leading to the same models, are common, e.g. Wilcox [34], Mohammadi and Pironneau [24], Chacon-Rebollo and Lewandowski [6]. URANS models can be viewed as approximating time averages

$$\overline{u}(x,t) := \frac{1}{\tau}\int_{t-\tau}^{t} u(x,t')dt' \text{ with fluctuation } u'(x,t) := (u-\overline{u})(x,t) \tag{1}$$

of solutions of the Navier-Stokes equations (NSE)

$$\nabla \cdot u = 0, u_t + u\cdot\nabla u - \nu\triangle u + \nabla p = f(x,t), \tag{2}$$

with the domain, kinematic viscosity, and initial and boundary conditions (BC-s) specified. There are a variety, 0-equation, 1-equation, 2-equation, and more-equation, of useful URANS models with (generally) increasing predictive ability as model complexity (e.g., number of equations and calibration parameters) increases. This report studies the extent flow statistics predicted by 1-equation models can be captured by a 1/2-equation model (derived in Section 2) which has 0-equation complexity.

The standard URANS approach is to model $\overline{u}(x,t)$ by eddy viscosity

$$v_t + v\cdot\nabla v - \nabla\cdot([2\nu+\nu_T]\nabla^s v) + \nabla q = \frac{1}{\tau}\int_{t-\tau}^{t} f(x,t')dt', \text{ and } \nabla\cdot v = 0.$$

Here $\nu_T = 0.55l\sqrt{k}$ is the eddy viscosity. The model representation of the turbulence length scale $l(x,t)$ and turbulent kinetic energy (TKE) $k(x,t) \simeq \frac{1}{2}\overline{|u'(x,t)|^2}$ must be specified. In Section 2 we show that with kinematic $l = \sqrt{2k}\tau$ the time evolution of the space-average of $k(x,t)$

$$k(t) = \frac{1}{|\Omega|}\int_\Omega k(x,t)dx \simeq \frac{1}{|\Omega|}\int_\Omega \frac{1}{2}\overline{|u'(x,t)|^2}dx.$$

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can be captured by a single ordinary differential equation (ODE) in time

$$\frac{d}{dt}k(t) + \frac{\sqrt{2}}{2}\tau^{-1}k(t) = \frac{1}{|\Omega|}\int_\Omega \nu_T|\nabla^s v|^2 dx. \tag{3}$$

Using $k(t)$ rather than $k(x,t)$ in ν_T reduces model complexity to that of a 0-equation model. Section 2.3 proves the positivity of $k(t)$ and the boundedness of the 1/2-equation model's kinetic energy and energy dissipation rate. Proposition 1 in Section 2.3 shows that when the time window τ is sufficiently small $k(t) \to 0$ (and thus ν_T also) reducing the model to the NSE. The other natural limit is whether the model solution converges to a Reynolds-averaged Navier-Stokes (RANS) approximation as $\tau \to \infty$. Analysis of this question is an open problem but our preliminary, heuristic analysis suggests it does hold.

The goal of URANS simulations is to give acceptable accuracy at a modest cost. One requirement for this is that the model's eddy viscosity does not over-dissipate. In Theorem 1, Section 2 we prove that for τ smaller than a specific value the model's time-averaged energy dissipation rate ϵ_{model} is bounded by the $\mathcal{O}(U^3/L)$ energy input rate

$$\lim_{T\to\infty} \sup \frac{1}{T}\int_0^T \epsilon_{model}(t)\, dt \lesssim (1 + \mathcal{R}e^{-1})\frac{U^3}{L}.$$

This proof of non-over dissipation is given for turbulence in a box in Section 2.4. Section 3 directly addresses accuracy, comparing 1/2-equation model velocity statistics with those of 1-equation models. Since simulations of the 1/2-equation model have reduced complexity compared to 1-equation models, the tests in Section 3 indicate that the 1/2-equation model's comparable accuracy makes it worthy of further study.

Related work. Finite time averaging (1) is one of the various averages, surveyed by Denaro [9], used to develop URANS models. We select it because it is analytically coherent and computationally feasible. The equation (3) is derived by space averaging the TKE equation developed by Prandtl [28] and Kolmogorov [21]. The equation for the spaced average TKE has the simpler form (3) due to the kinematic turbulence length $l = \sqrt{2k}\tau$, Section 2.2. This $l(x,t)$ was mentioned by Prandtl, Section 2.1, but developed much later. Our previous work [19], [20], [22] has found it to be effective when boundary layers are not primary and it has been used successfully by Teixeira and Cheinet [32], [33] in geophysical fluid dynamics (GFD) simulations. Our approach to 1/2-equation models is inspired by the pioneering work of Johnson and King [17], see also Wilcox [34] Section 3.7, Johnson [16]. This work captured variations of model parameters along a body or channel by deriving and solving an ODE in x, the streamwise direction. We also note that (1) means that there is not a sharp separation between our approach to URANS herein and time-filtered large eddy simulation (LES), reviewed in Pruett [30].

The analysis of energy dissipation rates in Section 2 builds on the important, fundamental, and compelling analysis of Doering and Foias [11] and Constantin and Doering [10]. A common failure mode of eddy viscosity models is over-dissipation (even predicting a laminar solution). Thus energy dissipation analysis, inspired by [11, 10], directly addresses the practical issues of turbulence modeling, e.g. [19, 20, 23].

2. The 1/2-equation model

The 1-equation model is reviewed in Section 2.1 followed by the derivation of the 1/2-equation model studied herein in Section 2.2. Analytical properties of the