INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 22, Number 2, Pages 157–177

doi: 10.4208/ijnam2025-1008

WEAK GALERKIN FINITE ELEMENT METHOD BASED ON POD FOR NONLINEAR PARABOLIC EQUATIONS

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Abstract. In this paper, we establish a novel reduced-order weak Galerkin (ROWG) finite element method for solving parabolic equation with nonlinear compression coefficient. We first present the classical weak Galerkin finite element discretization scheme and derive the optimal error estimates. Then we apply a proper orthogonal decomposition (POD) technique to develop the ROWG method, which can effectively reduce degrees of freedom and CPU time. The optimal order error estimates are also derived, and the algorithm flow is provided. Finally, some numerical experiments illustrate the performance of the ROWG method. The numerical results show that the proposed ROWG method is efficient for solving nonlinear parabolic equations.

Key words. Weak Galerkin finite element method, nonlinear parabolic equations, proper orthogonal decomposition.

1. Introduction

In this paper, we consider the following parabolic equations with nonlinear compression coefficient:

$$g(u)u_t - \nabla \cdot (D\nabla u) = f, \quad (x,t) \in \Omega \times J, \tag{1a}$$

$$u = u_0, \quad (x,t) \in \Omega \times \{t=0\}, \tag{1b}$$

$$u = \phi, \quad (x,t) \in \partial\Omega \times J, \tag{1c}$$

where Ω is a polygonal region in $\mathbb{R}^2$ with Lipschitz continuous boundary. Here D is a symmetric positive definite matrix, $g(u)$ is a sufficiently smooth function with bounded derivatives up to the second-order, and there exist two constants g_*, g^* such that

$$0 \le g_* \le g(u) \le g^*, \quad \|u\|_\infty < \infty.$$

And the assumptions that the solution of (1) satisfies can be found in the literature [4].

The weak formulation of (1) is to find $u \in H^1(\Omega)$ such that

$$(g(u)u_t, v) + (D\nabla u, \nabla v) = (f, v), \quad \forall v \in H_0^1(\Omega). \tag{2}$$

The weak Galerkin (WG) finite element method is first proposed in [14, 15, 12]. It can be viewed as an extension of the standard finite element method. The key to WG method is the introduction of weak functions and weak gradients. In comparison with conventional finite element method (FEM), the WG method has higher robustness in boundary processing and is more suitable for grids with hanging points. In recent years, the WG method has been widely used to solve the Darcy-Stokes equation [3, 8], quasi-linear elliptic problems [18, 2], etc.

The proper orthogonal decomposition (POD) technique has been combined with the finite element method since 2001 and successfully applied to solve parabolic equations [7]. This method uses several layers of images to perform a low-dimensional

Received by the editors on June 7, 2024 and, accepted on September 21, 2024.
2000 *Mathematics Subject Classification.* 78M10.
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approximation of the piecewise polynomial function space. In this way, a new finite element function space is constructed. Within the allowed error range, POD can effectively transform the original high-dimensional model into a low-dimensional one, which significantly improves the computational efficiency. In recent years, researchers have been actively exploring the combination of POD and different types of numerical methods, including the finite difference method, the finite volume method, and the hybrid finite element method [1, 5, 6, 9], etc. Very recently, Zhang el at. in [16] considers the formulation and theoretical analysis of a reduced-order numerical method constructed by POD for nonlocal diffusion problems. Zhao et al. [17] first linked POD with weak Galerkin finite element method, but only gave the algorithm flow without the corresponding theoretical analysis.

In this paper, we apply the POD technique to develop a novel reduced-order weak Galerkin (ROWG) finite element method [17, 10, 13] for solving the nonlinear parabolic problem (1). We construct a new correlation matrix and provide convergence analysis under the L^2 and the discrete H^1 norms. The rest of the paper is organized as follows: In Section 2, we first introduce the concepts of discrete weak functions and weak derivatives for WG method. Then we establish the fully discrete WG scheme for problem (1), and derive the optimal error estimates. In Section 3, we construct the POD basis and build the fully discrete ROWG scheme. The optimal error estimates for the ROWG scheme are presented, and the algorithm process is shown. In Section 4, we give some numerical examples and compare the CPU time of the ROWG scheme and the WG scheme for all examples. Conclusions are given in Section 5.

2. Classical WG method

In this section, we consider the following discrete weak Galerkin finite element space $WG(P_r, P_{r-1}; P_{r-1}^2)$. Let $\mathcal{T}_h$ be a partition of Ω that satisfies the conditions in [11]. We denote

$$V_h = \Big\{ v = \{v_0, v_b\} : v_0|_K \in P_r(K), v_b|_e \in P_{r-1}(e), e \subset \partial K, K \in \mathcal{T}_h \Big\}, \tag{3}$$

and its subspace V_h^0 as

$$V_h^0 = \Big\{ v \in V_h : v_b|_{\partial\Omega} = 0 \Big\}. \tag{4}$$

For any $v \in V_h$, its weak gradient $\nabla_\omega v$ satisfes that

$$(\nabla_\omega v, \phi)_K = -(v_0, \nabla \cdot \phi)_K + \langle v_b, \phi \cdot \mathbf{n} \rangle_{\partial K} \quad \forall \phi \in P_{r-1}^2(K), \tag{5}$$

Let $t_n = n\Delta t\, (n = 1, 2, \cdots, N)$, $\Delta t = T/N$, and denote $u^n = u(t_n)$. The fully discrete WG finite element scheme for (1) is to find $U^n = \{U_0^n, U_b^n\} \in V_h$ such that

$$\left(g(U^{n-1}) \frac{U^n - U^{n-1}}{\Delta t}, v \right) + a_s(U^n, v) = (f^n, v), \quad \forall v = \{v_0, v_b\} \in V_h^0, \tag{6}$$

with the initial value $U^0 = Q_h u^0$. Here the bilinear form

$$a_s(u, v) = \sum_{K \in \mathcal{T}_h} (D\nabla_\omega u, \nabla_\omega v)_K + h_K^{-1} \langle Q_b u_0 - u_b, Q_b v_0 - v_b \rangle_{\partial K}.$$

We define $|||v||| = \sqrt{a_s(v, v)}$ and $\|v\|_h = \left(\sum_{K \in \mathcal{T}_h} \|\nabla v_0\|_T^2 + h_T^{-1} \|Q_b v_0 - v_b\|_{\partial K}^2 \right)^{\frac{1}{2}}$ on V_h, and $|||\cdot|||$ is equivalent to $\|\cdot\|_h$ (cf. [12]).

We give the estimate of the errors between WG solution and the analytical solution. Denote $\theta_h^n = E_h u^n - U^n$, $\eta_h^n = u^n - Q_h u^n$, $\tau_h^n = Q_h u^n - E_h u^n$, where