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THE NAVIER-STOKES-ω/NAVIER-STOKES-ω MODEL FOR FLUID-FLUID INTERACTION USING AN UNCONDITIONALLY STABLE FINITE ELEMENT SCHEME

QINGHUI WANG, PENGZHAN HUANG*, AND YINNIAN HE

Abstract. In this article, for solving fluid-fluid interaction problem, we consider a Navier-Stokes-ω/Navier-Stokes-ω model, which includes two Navier-Stokes-ω equations coupled by some nonlinear interface conditions. Based on an auxiliary variable, we propose a fully discrete, decouple finite element scheme. We adopt the backward Euler scheme and mixed finite element approximation for temporal-spatial discretization, and explicit treatment for the interface terms and nonlinear terms. Moreover, the proposed scheme is shown to be unconditionally stable. Then, we establish error estimate of the numerical solution. Finally, with a series of numerical experiments we illustrate the stability and effectiveness of the proposed scheme and its ability to capture basic phenomenological features of the fluid-fluid interaction.

Key words. Navier-Stokes-ω model, fluid-fluid interaction, auxiliary variable, unconditional stability.

1. Introduction

In many scientific fields and practical applications, numerical simulation is an important aspect for multi-physics and multi-domain of one immiscible fluid with another fluid, such as simulations of atmosphere-ocean interaction problem [1, 2, 3, 4] in environmental engineering and so on. Herein, we will consider a fluid-fluid interaction problem with some nonlinear interface conditions, which are known as the rigid-lid condition (e.g. approximate the atmosphere-ocean surface as flat). It allows for the energy to transfer back and forth across the interface, while the global energy of the system remains conserved [5].

At the time of writing, numerous works are devoted to fluid-fluid interaction models with nonlinear interface condition [6, 7]. Bernardi et al. [8, 9, 10] have studied two immiscible turbulent fluids on adjacent subdomains, but it limits the effectiveness of calculations because of a large and decoupled system. Connors et al. [11] have proposed two decoupled time stepping methods based on the partitioned time stepping methods. One of them is the geometric averaging method, whose key benefit is the unconditional stability. This method has further developed in [12, 13, 14, 15, 16, 17]. Recently, Aggul et al. [18] have proposed a large eddy simulation with correction model, which used the defect correction to control efficiently the model error.

When at least one of the flow enters the turbulent state in subdomain, we need to choose a turbulence model. Recently, the approximate deconvolution model of turbulence is considered for fluid-fluid interaction [19]. In this paper, we will consider the Navier-Stokes-ω model, which has non-filtered velocity on the interface. In fact, Layton et al. [20] showed that the discrete Navier-Stokes-ω simulation had greater accuracy at less cost and required significantly fewer degrees of freedom than a comparable Navier-Stokes-α simulation. In [21], the authors proved existence and

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*Corresponding author.

uniqueness of strong solutions of Navier-Stokes-ω model and gained convergence to a weak solution of the Navier-Stokes equations as the averaging radius decreased to zero. Furthermore, Manica et al. [22] used van Cittert approximate deconvolution to improve accuracy and Scott-Vogelius elements to provide pointwise mass conservative solutions, and removed the dependence of the Bernoulli pressure error on the velocity error for Navier-Stokes-ω model. Recently, with help of the geometric averaging approach, Aggul et al. [23] considered the Navier-Stokes-ω model for fluid-fluid interaction problem, which was showed to be unconditionally stable. Further, based on large eddy simulation with correction, the authors designed a second-order scheme in time [18].

In this article, we will design a fully explicit treatment to decouple nonlinear interface conditions and also obtain the unconditional stability. From the point of view of numerical discretization for partial differential equations, it is natural to hope that nonlinear terms can be treated explicitly. In the meantime, the unconditional stability will be kept. The scalar auxiliary variable has been initially applied to the gradient flow problem in [24, 25], which can keep unconditional energy stability. Inspired by this idea, Li et al. [26] have solved the magnetohydrodynamic by an exponential scalar auxiliary variable method. Jiang and Yang [27] have developed two decoupled ensemble schemes for the Stokes-Darcy system, by combining the scalar auxiliary variable idea with the ensemble time step method. Additionally, Li et al. [28] have further established an unconditionally energy stable finite element scheme for a nonlinear fluid-fluid interaction model. A generalized scalar auxiliary variable approach has been considered for the Navier-Stokes-ω/Navier-Stokes-ω equations based on the grad-div stabilization [29].

In current work, we introduce an auxiliary variable in exponential function to deal with the Navier-Stokes-ω/Navier-Stokes-ω model, where nonlinear terms and nonlinear interface terms are treated explicitly. The proposed scheme enjoys the following features: (i) it is unconditionally stable; (ii) unlike the geometric averaging scheme, explicit treatment is considered for the interface terms. The structure of this paper is arranged as follows: In Section 2, we will introduce some notations and function spaces. In Section 3, the Navier-Stokes-ω/Navier-Stokes-ω model is showed. Moreover, a fully discrete finite element scheme based on an auxiliary variable is designed, and unconditional stability is obtained. Section 4 develops the theory for the scheme, showing analysis of convergence. In Section 5, we use several numerical experiments to test the stability and convergence of the proposed scheme, and to show its ability to capture basic phenomenological features of the fluid-fluid interaction. We close with some concluding remarks in Section 6.

2. Preliminary

This section summarizes some necessary notations and inequalities.

Consider the spatial domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ that consists of two subdomains Ω_1 and Ω_2 coupled across their shared interface $I \subsetneqq \partial\Omega$. Next, $\|\cdot\|_0$ and $(\cdot, \cdot)$ are represented as $L^2(\Omega_i)$ $(i = 1, 2)$ norm and its inner product. Additionally, $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W^{m,p}}$ are expressed as the Lebesgue space $L^p(\Omega_i)$ norms and the Sobolev space $W^{m,p}(\Omega_i)$ norms for $m \in \mathbb{N}^+$, $1 \leqslant p \leqslant \infty$. Besides, for X_i being a normed function space in Ω_i, $L^p(0, T, X_i)$ is the space of all functions defined on $[0, T] \times \Omega_i$ and it's norm represents

$$\|\mathbf{u}\|_{L^p(0,T,\mathbf{X}_i)} = \left(\int_0^T \|\mathbf{u}\|_{X_i}^p dt \right)^{\frac{1}{p}}, \ p \in [1, \infty).$$