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ANALYSIS OF AN EFFICIENT PARAMETER UNIFORM DOMAIN DECOMPOSITION APPROACH FOR SINGULARLY PERTURBED GIERER-MEINHARDT TYPE NONLINEAR COUPLED SYSTEMS OF PARABOLIC PROBLEMS

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Abstract. This article presents an efficient domain decomposition algorithm of Schwarz waveform relaxation type for singularly perturbed Gierer-Meinhardt type nonlinear coupled systems of parabolic problems where the diffusion terms in each equation are multiplied by small parameters of different magnitudes. The magnitude of these small parameters leads to the sharpness and boundary layer behavior in the solution components. Our present algorithm considers a suitable decomposition of the domain and decouples the process of approximating the solution components at each time level. There are two different schemes proposed in this work. Specifically, the schemes use the backward Euler method combined with a suitable component-wise splitting for time discretization, while employing the central difference scheme for spatial discretization. The two numerical schemes differ in their splitting methods: Scheme 1 employs a Jacobi-type split, whereas Scheme 2 utilizes a Gauss-Seidel-type split. The exchange of information between neighboring subdomains is achieved through piecewise-linear interpolation. The convergence analysis of the algorithm is demonstrated using some auxiliary nonlinear systems. It is shown that the present procedure provides uniformly convergent numerical approximations to the solution having sharp spike components. Numerical experiments demonstrate that the considered algorithm with present discretization is more efficient in terms of accuracy and iteration counts than with the standard available approaches.

Key words. Schwarz waveform relaxation, domain decomposition method, Gierer-Meinhardt systems, singular perturbation, efficient algorithms, computational cost, CPU time.

1. Introduction

In the past two decades, analysis and simulation of coupled systems of parameterized nonlinear problems with boundary and interior layer phenomena have drawn the attention of physicists and applied mathematicians due to their numerous appearances inside the scientific community, see for e.g. nonlinear problems in [1, 2, 3, 4]. The computational analysis of these nonlinear PDE systems is not straightforward as it involves perturbation parameters of different magnitudes and coupling in the nonlinear terms [5, 6]. These problems are characterized by singularly perturbed problems when the arbitrary small perturbation parameters can not be approximated by zero. Gierer-Meinhardt model is one of the activator-inhibitor types of singularly perturbed nonlinear reaction diffusion system, which appears in pattern formation and morphogenesis and is of singularly perturbed multiple scale nature. Here, small diffusion of the activator and large diffusion of the inhibitor lead to sharp spike at the boundary points and make it of multiple-scale nature [7].

Veerman and Doleman [3] studied the existence and stability of localized pulses on the GM (Gierer-Meinhardt) equation with a slow nonlinearity, which can be

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formulated by the following multiple-scale reaction-diffusion equations

$$\left\{\begin{array}{l}\partial_t y_1 = \partial_x^2 y_1 + F(y_1, y_2),\\ \partial_t y_2 = \varepsilon^2 \partial_x^2 y_2 + G(y_1, y_2),\end{array}\right. \tag{1}$$

where $0 < \varepsilon \ll 1$ is a small parameter. A rigorous existence and stability analysis for GM equation corresponding to interior spike, boundary spike and two boundary spikes are examined in [8, 9] based on the boundary conditions. In general, the procedures of deriving wellposedness of a nonlinear problem use the fixed point theory, as given in [10].

Hence, we consider the following generalized version of nonlinear coupled system of GM equation for further behavior of its solutions and their computational efficiency, on $\Omega = \mathfrak{D} \times (0, \mathcal{T}]$, where $\mathfrak{D} = (0, 1)$:

$$\boldsymbol{\mathcal{L}} \boldsymbol{y} := \partial_t \boldsymbol{y}(x,t) - \boldsymbol{\varepsilon} \partial_x^2 \boldsymbol{y}(x,t) + \boldsymbol{f}(x,t,\boldsymbol{y}) = 0, \tag{2}$$

with initial and boundary conditions

$$\boldsymbol{y}(x,0) = \boldsymbol{\phi}(x),\ 0 \le x \le 1,$$

$$\boldsymbol{y}(0,t) = \boldsymbol{\varphi}_0(t),\ \boldsymbol{y}(1,t) = \boldsymbol{\varphi}_1(t),\ 0 < t \le \mathcal{T}.$$

Here, $\boldsymbol{\varepsilon} = diag(\varepsilon_1, \varepsilon_2)$ is a diagonal matrix such that the parameters ε_1 and ε_2 can have different magnitudes with $0 < \varepsilon_1 \le \varepsilon_2 \le 1$; $\boldsymbol{\mathcal{L}} = (\mathcal{L}_1, \mathcal{L}_2)^T$, the solution $\boldsymbol{y} = (y_1, y_2)^T$, the boundary data $\boldsymbol{\varphi} = (\varphi_1, \varphi_2)^T$, and initial data $\boldsymbol{\phi} = (\phi_1, \phi_2)^T$. We consider that the functions $\boldsymbol{f}, \boldsymbol{\phi}, \boldsymbol{\varphi}_0$, and $\boldsymbol{\varphi}_1$ are sufficiently smooth and appropriate compatibility conditions hold for the present problem, see [11]. Further, for all $(x,t,\boldsymbol{y}) \in \overline{\Omega} \times \mathbb{R}^2$, assume that the nonlinear reaction term $\boldsymbol{f} = (f_1, f_2)^T$ satisfies

$$\frac{\partial f_s}{\partial y_s}(x,t,y_1,y_2) \ge \overline{\alpha} > 0,\ \frac{\partial f_s}{\partial y_q}(x,t,y_1,y_2) \le 0,\ s \ne q,$$

$$\sum_{q=1}^{2} \frac{\partial f_s}{\partial y_q}(x,t,y_1,y_2) \ge \alpha > 0,\ s = 1,2.$$

The investigation of multiple scale singularly perturbed problems became significantly important due to the rapid variation of continuous solutions in specific layer regions. Conventional simulators to solve such problems often encounter challenges in effectively handling sufficiently small diffusion terms. Consequently, the need arises for robust numerical methods to address these issues [12, 13]. Fitted/graded mesh [14, 15, 16, 17, 18, 19], fitted operator [20], and domain decomposition approaches are often used to construct such robust numerical algorithms for computational efficiency. This paper utilizes the domain decomposition approach, which is frequently used to reduce the computational cost, as like splitting approaches for coupled systems given in [21].

Domain decomposition approaches have gained significant popularity for effectively solving partial differential equations [22, 23, 24, 25, 26, 27]. Among these methods, Schwarz Waveform Relaxation (SWR) stands out as a special class specifically designed for time-dependent problems and was initially introduced in [29, 28]. The SWR approach involves dividing the original domain into space-time subproblems, where each subproblem is solved over the entire time window independently before exchanging interface data between subdomains. Notably, each subdomain can adopt its own space and time grid along with discretization techniques, making SWR highly adaptable and well-suited for parallel computing. SWR methods have been developed for a wide range of regular and singularly perturbed time-dependent