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AN UNCONDITIONALLY ENERGY-STABLE SAV-DG NUMERICAL SCHEME FOR TUMOR GROWTH MODEL

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Abstract. In this paper, we propose a linear, fully decoupled and unconditionally energy-stable discontinuous Galerkin (DG) method for solving the tumor growth model, which is derived from the variation of the free energy. The fully discrete scheme is constructed by the scalar auxiliary variable (SAV) for handling the nonlinear term and backward Euler method for the time discretization. We rigorously prove the unconditional energy stability and optimal error estimates of the scheme. Finally, several numerical experiments are performed to verify the energy stability and validity of the proposed scheme.

Key words. Tumor growth model, DG method, SAV approach, optimal error estimates.

1. Introduction

According to the International Agency for Research on Cancer (IARC), the malignant tumor has become one of the major diseases affecting human health in the world [1–4]. Due to the lack of understanding the mechanisms of tumor growth, many difficulties have been encountered in the treatment process [5, 6]. The research of tumor growth holds significance for clinical therapy, thereby generating considerable interest from the fields of medicine, genetics and biology [7–11]. As we know, an accurate mathematical model will help medical staff to better understand the mechanism of tumor growth [12–15].

Recently, many researchers have focused on the mathematical models for investigating the tumor growth [16–22]. In order to better predict the evolution of tumor, the phase-field model has been widely used to study the tumor growth [23–28]. In [23] the authors used the Cahn-Hilliard equation to describe multispecies tumor growth and tumor-induced angiogenesis. In [24], the Cahn-Hilliard type equation with degenerate mobility had been used to simulate the evolution and growth of solid tumors. In [27] the authors introduced the Allen-Cahn equation to describe the progression of tumoral expansion within the growth region. In this study, we derive the tumor growth model based on the law of energy conservation in the polygonal domain Ω in $\mathbb{R}^d$ $(d = 2, 3)$. The total free energy of the system is defined as

$$E(\phi,\sigma) = \int_\Omega [F(\phi) + \frac{\lambda}{2}|\nabla\phi|^2 + \frac{\alpha}{2}\phi^2 + \frac{\beta}{2}|\nabla\sigma|^2 - (\chi\phi - \frac{\gamma}{2}\sigma + s)\sigma]\ d\boldsymbol{x}, \tag{1}$$

with the following double-well type potential

$$F(\phi) = \frac{16}{\varepsilon}\phi^2(1-\phi)^2,$$

where $\varepsilon > 0$ is the time scale. By using gradient flow method [29–31], the following equation can be derived as

$$\phi_t = -\frac{\delta E}{\delta\phi} = \lambda\Delta\phi - f(\phi) + \chi\sigma - \alpha\phi, \qquad \text{in } \Omega\times(0,T), \tag{2a}$$

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$$\sigma_t = -\frac{\delta E}{\delta \sigma} = \beta\Delta\sigma + s + \chi\phi - \gamma\sigma, \qquad \text{in } \Omega\times(0,T), \tag{2b}$$

subject with the following initial and the boundary conditions

$$\phi(\boldsymbol{x},0) = \phi_0,\ \sigma(\boldsymbol{x},0) = \sigma_0, \qquad \text{in } \Omega, \tag{3a}$$

$$\partial_{\boldsymbol{n}}\phi(\boldsymbol{x},t) = 0,\ \partial_{\boldsymbol{n}}\sigma(\boldsymbol{x},t) = 0, \qquad \text{on } \partial\Omega, \tag{3b}$$

where $f(\phi) = F'(\phi)$ and $\boldsymbol{n}$ is the unit outward normal vector on $\partial\Omega$. The unknown functions ϕ and σ are the phase function and the concentration of the nutrient, respectively. The physical parameters λ represents the phase field diffusion coefficient, χ denotes the rate of tumor growth, α is the rate of normal cell apoptosis, β represents the nutrient diffusion coefficient, s denotes the sustained supply of the nutrient, and γ is the natural decay of the nutrient, which are positive.

To the best of the author's knowledge, most of the researches related to tumor growth model have focused on numerical simulations and numerical analysis [23–28, 32–38]. Lorenzo et al. [33] used isometric analysis to solve the Allen-Cahn equation and simulated the tumor growth process. In what follows, Mohammadi et al. [34,35] also simulated the same model numerically by applying the finite difference method and the meshless method. For describing the growth dynamics of avascular tumors, Medina et al. developed a hybrid discontinuous Galerkin numerical scheme for the Cahn-Hilliard equation in [36]. However, the above-mentioned researches lack the stability and optimal error estimates of the coupled model. Agosti et al. [24] employed the finite element method to discrete the Cahn-Hilliard equation, and proved the existence and uniqueness of the proposed semi-discrete scheme, without the convergence analysis. In [37], Xu et al. used the BDF1 method to construct a linear, fully decoupled and energy stable numerical scheme for the Cahn-Hilliard-Navier-Stokes system, and derived the optimal error estimates. However, it should be noted that the proposed scheme is semi-discrete in time. For the Cahn-Hilliard-Brinkman-Ohta-Kawaski tumor growth model, the authors in [38] proposed a fully discrete numerical scheme with the H^1-norm optimal error estimates by using the discontinuous Galerkin (DG) method. They verified the validity and stability of the proposed scheme through several numerical experiments.

Note that constructing an efficient numerical scheme for the tumor growth model is definitely not a simple task due to highly nonlinear term and strongly coupled term. First of all, the main difficulty caused by the nonlinear term in model (2) is how to propose an effective energy stable time discrete strategy for solving the corresponding phase field equation. We use the scalar auxiliary variable (SAV) method [39–44] to deal with the nonlinear term and transform the original equation into an equivalent linear system. Furthermore, the diffusion term in the phase field equation produces a discontinuity at the phase transition boundary, and the discontinuous Galerkin (DG) method [45–54] performs well for the complex boundary problems. In this paper, we develop a fully discrete DG scheme which is linear, full decoupled and unconditionally energy stable. In addition, the unique solvability and the L^2-norm optimal error estimates are given in details. Finally, the stability and validity of the proposed scheme is verified by a series of numerical experiments.

The outline of this paper is organized as follows. Section 2 demonstrates that the tumor growth model is unconditionally energy dissipative. In Section 3, we develop a linear, decoupled fully discrete numerical scheme, and derive the unique solvability and discrete energy law. The prove of optimal error estimates for the fully discrete scheme is given in Section 4. In Section 5, we present some numerical