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SCHWARZ METHOD IN A GEOMETRICAL MULTI-SCALE DOMAIN WITH CONTINUOUS OR DISCONTINUOUS JUNCTIONS

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Abstract. A model parabolic linear partial differential equation in a geometrical multi-scale domain is studied. The domain consists of a two-dimensional central node, and several one-dimensional outgoing branches. The physical coupling conditions between the node and the branches are either continuity of the solution or continuity of the normal flux. An iterative Schwarz method based on Robin transmission conditions is adjusted to the problem in each case and formulated in substructured form. The convergence of the method is stated. Numerical results when the method is used as preconditioner for a Krylov method (GMRES) are provided.

Key words. Finite volume scheme, parabolic problem, multi-scale domain, domain decomposition, stability and convergence of numerical methods, Schwarz methods, Robin interface condition.

1. Introduction

Geometrical multi-scale problems involve coupling dimensionally-heterogeneous partial differential equations. These are generally reduced models of reality for which computational costs are much lower, although the full-dimensional models have to be kept in the neighborhoods of the bifurcations or junctions. Such dimensionally-heterogeneous problems have been studied in different fields. In some problems of this type, such as fluid flow in a fracture in a porous medium, or blood flow in a small vessel in biological tissue, the part of the domain whose geometric dimension is reduced is included in a part of the domain whose original dimension is retained [30, 15]. In this paper, we place ourselves in a context where parts of the domain of different dimensions are side by side. In [28, 29], where domains of different geometric dimensions are juxtaposed, domain decomposition methods such that the interfaces take place at the frontier between the domains of different dimensionalities are used. The partitioning methodology takes full advantage of the small number of interface unknowns (which is not the case with an embedded model). In [3], a general theoretical framework, which generalizes this methodology, is developed by recasting the variational formulation in terms of coupling interface variables. The last papers [3, 28, 29] present examples in the fields of heat transfer, linear elasticity, hydraulic networks.

Here, we consider a model problem posed in a domain which derives from a network of rods. The rod structures are some connected unions of cylinders. They are for instance systems of pipes in industrial installations or canal systems. Often, rods are considered as one-dimensional (1D) at some distance from the junctions. This is particularly true for modeling the human blood circulatory system [5, 37].

Continuing a study whose results are presented in [33, 34, 35, 36, 41], we consider here the heat equation set in a geometrical multi-scale domain called D_ε that derives from a two-dimensional (2D) thin rod structure Ω_ε. This rod structure itself derives from a very simple graph that is a single bundle with one node O from which p

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edges, $e_j = [O, O_j], j = 1, ..., p$, depart. The length of the edge $|e_j| = l_j, j = 1, ..., p$. The axis Ox^{e_j} has the direction of $[OO_j)$ and a local Cartesian coordinate systems $(O, x^{e_j}, y^{e_j}), j = 1, ..., p$, is considered. The thin rod structure is an union of p thin rectangles to which a bounded domain ω_0^ε is added at the center to smooth out the corners. Each rectangle being carried by a single edge, the rectangle on edge e_j is defined by $\{(x^{e_j}, y^{e_j}) \in I\!\!R^2 : x^{e_j} \in (0, l_j), y^{e_j} \in (-\theta_j\varepsilon/2, \theta_j\varepsilon/2)\}, j = 1, ..., p$, where $\varepsilon > 0$ is a small parameter and $\theta_j, j = 1, ..., p$, are positive numbers independent of ε. We build D_ε from Ω_ε, keeping a small 2D area around the node as it is, which we call $\Omega(0)$, and assuming that each rectangular branch from this area is now reduced to the 1D central edge around which it is built. To be more precise, let $\delta > 0$, $\Omega(0)$ is the truncated at the distance δ from the junction part of Ω_ε containing the junction. Ω_ε and D_ε can be seen from Figure 1.

The segments $\gamma_j, j = 1, ..., p$, are the 1D-2D interfaces at $x^{e_j} = \delta$ in D_ε and from the above $|\gamma_j| = \theta_j\varepsilon$. The part of the jth rectangular branch that is reduced to dimension 1 in space is denoted S_j. Even if the dependence of δ versus ε is not the subject of this paper, remember that if δ is of order $\varepsilon|ln(\varepsilon)|$ and if the source term is assumed to only depend on the longitudinal variables, then it was proved by asymptotic analysis that the solutions of the heat equation on D_ε and Ω_ε are very close to each other (for more details see Theorem 6.2 in [35] and Theorem 3.3 in [36]). This justifies to work in D_ε.

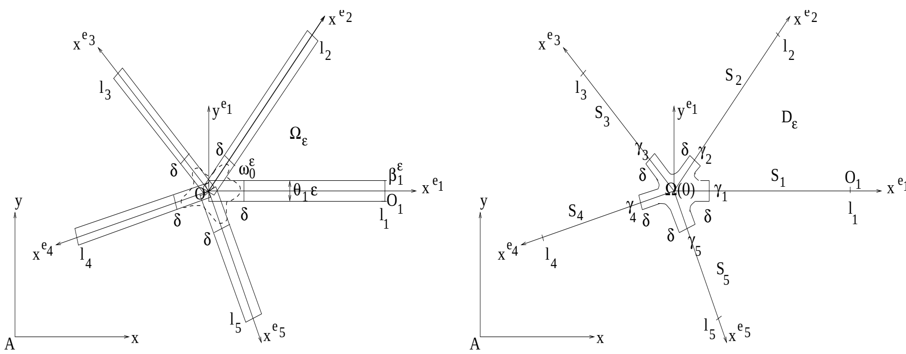

FIGURE 1. The rod structure Ω_ε on the left and the geometrical multi-scale domain D_ε on the right.

Two options are provided for the 1D-2D physical coupling conditions on the interfaces $\gamma_j, j = 1, ..., p$: the solution and the mean value of the flux are continuous, or the flux and the mean value of the solution are continuous. First, monolithic finite volume numerical scheme of hybrid dimension was defined in [35] and [41] in either case. These schemes lead to linear systems that give the volume solution of the model problem and they are solved by a direct method. Second, iterative substructuring domain decomposition methods used as preconditioner for a Krylov method (GMRES, see [40]) and such that the interfaces take place at γ_j were also used. They result in an interface system at each time step, which depends on the 1D and 2D separated problems (we are not here in the waveform relaxation setting described in [24]). However, the boundary conditions chosen to define the 2D separated problem were mainly pointwise Dirichlet conditions when the solution is continuous, or pointwise Neumann conditions when the flux is continuous.