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SPLITTING SCHEMES FOR BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. This paper concerns splitting methods for solving backward stochastic differential equations (BSDEs). By splitting the original d-dimensional BSDE into d BSDEs and approximating these split BSDEs, we propose splitting schemes for the BSDE. The splitting schemes are rigorously analyzed and first-order error estimates are theoretically obtained. Numerical tests are given to verify the theoretical results.

Key words. Backward stochastic differential equations, splitting method, splitting scheme, error estimate.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a complete, filtered probability space with $\mathbb{F} = \{\mathcal{F}_t\}_{0\leq t\leq T}$ being the natural filtration generated by a d-dimensional Brownian motion $W_t = (W_t^1, W_t^2, \cdots, W_t^d)^\top$. The general form of backward stochastic differential equation (BSDE) on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ is

$$Y_t = \varphi(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad t \in [0, T], \tag{1}$$

where $X_t = X_0 + W_t$ is a forward diffusion process with $X_0 \in \mathcal{F}_0$ being the initial condition; $T > 0$ is the deterministic terminal time; $f : [0, T] \times \mathbb{R}^d \times \mathbb{R}^p \times \mathbb{R}^{p\times d} \to \mathbb{R}^p$ and $\varphi : \mathbb{R}^d \to \mathbb{R}^p$ are the generator and the terminal function of the BSDE, respectively. We note that the integral term with respect to W_s of the BSDE is the Itô-type integral. The pair of processes $(Y_t, Z_t) : [0, T] \times \Omega \to \mathbb{R}^p \times \mathbb{R}^{p\times d}$ is called an L^2-adapted solution of the BSDE (1) if it is $\mathcal{F}_t$-adapted, square integrable, and satisfies (1).

In 1990, under certain standard conditions, Pardoux and Peng [26] originally proved the existence and uniqueness of the solutions of general nonlinear BSDEs. In 1991, Peng [29] found the nonlinear Feynman-Kac formula, that is, under some regularity conditions, the solution (Y, Z) of (1) can be represented as

$$Y_t = u(t, X_t), \quad Z_t = \nabla_x u(t, X_t), \quad t \in [0, T), \tag{2}$$

where $u : [0, T] \times \mathbb{R}^d \to \mathbb{R}^p$ is the classical solution to the following second-order parabolic partial differential equation (PDE)

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} + f(t, x, u, \nabla_x u) = 0, \quad (t, x) \in [0, T) \times \mathbb{R}^d \tag{3}$$

with the terminal condition $u(T, x) = \varphi(x)$ for $x \in \mathbb{R}^d$. The representation (2) deeply connects the BSDE and the parabolic PDE, which enables us to develop numerical schemes for the BSDE (1) by solving the associated parabolic PDE (3), and vice versa. Since then, significant efforts have been made to study BSDEs

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due to their important applications in various fields, such as mathematical finance, PDEs, stochastic control, risk measurement, game theory, deep learning and so on (see, e.g., [9, 23, 28, 29, 13] and references therein).

It is usually difficult to obtain the analytical solutions of BSDEs, and thus numerical methods for solving BSDEs are in high demand. Recently, a lot of work has been put into developing effective numerical methods for solving BSDEs. One of the most widely used approaches is to discretize BSDEs directly, leading to various discretization schemes for solving BSDEs [1, 2, 6, 16, 21, 36, 3]. Popular temporal discretization strategies include Euler-type methods [14, 15, 35], the generalized θ-schemes [31, 36, 38], Runge-Kutta schemes [5], the multistep schemes [4, 19, 20, 30, 37, 39, 40], strong stability preserving multistep schemes [10, 11], and extrapolation methods [32, 33], etc. Additionally, there are several numerical schemes for solving BSDEs proposed based on the nonlinear Feynman-Kac formula and the numerical solutions of parabolic PDEs associated with BSDEs, such as in the papers [7, 22, 24].

Parabolic equations have found wide applications in various fields, such as the heat transfer in a superconductor, the chemical reaction from chemical engineering and the modeling of economic processes, etc. Thus various numerical methods have been developed for solving parabolic PDEs [8, 25]. For multi-dimensional PDEs, splitting methods, including alternating direction implicit (ADI) [8, 27] and locally one-dimensional (LOD) [34] methods, demonstrated significant advantages due to their low computational complexity and high computational efficiency. The LOD method, also known as fractional step methods, and the ADI method solve a multi-dimensional equation by converting the multi-dimensional equation to a succession of one-dimensional equations, while providing accurate numerical solutions. These methods have been extended to solve nonlinear PDE problems and other physical problems. However, up to now, there are still very few research on splitting methods for solving BSDEs.

To fill the gap, in this paper, we shall propose splitting schemes for solving BSDEs. To obtain the splitting schemes, on each time subinterval $[t_n, t_{n+1}]$ of a given time partition $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$, first we split the BSDE (1) into d BSDEs with their solutions $\{(\bar{Y}_t^i, \bar{Z}_t^i), i = 1, \cdots, d\}$, then, by approximating these split BSDEs, we construct our splitting schemes for solving the original BSDE (1). The main advantage of the schemes is that only one-dimensional approximations are required to calculate the conditional mathematical expectations, which may reduce computational cost. We rigorously provide the theoretical error estimates, which show the first-order convergence rate of the schemes. Our numerical tests also validate our theoretical results, and show the accuracy and effectiveness of our splitting schemes.

The rest of this paper is organized as follows. In Section 2, we propose splitting schemes for solving BSDEs by splitting the original d-dimensional BSDE into d BSDEs and approximating these split BSDEs. In Section 3, we rigorously prove the first-order convergence rate in time for the splitting schemes. Several numerical tests are presented to show the accuracy and effectiveness of our splitting schemes in Section 4. In Section 5, the conclusions are given.

2. Splitting methods for BSDEs

First, we introduce some notations. Use $\Delta W_{t,s}$ to denote the increment $W_s - W_t = (\Delta W_{t,s}^1, \cdots, \Delta W_{t,s}^d)^\top$ of the Brownian motion W_s for $s \geq t$, where $\Delta W_{t,s}^i =$