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THE WEIGHTED AND SHIFTED TWO-STEP BDF METHOD FOR ALLEN-CAHN EQUATION ON VARIABLE GRIDS

MINGHUA CHEN*, FAN YU, AND QINGDONG ZHANG

Abstract. The weighted and shifted seven-step BDF method is proposed by the authors [Akrivis, Chen, and Yu, IMA. Numer. Anal., DOI:10.1093/imanum/drae089] for parabolic equation on uniform meshes. In this paper, we study the weighted and shifted two-step BDF method (WSBDF2) for the Allen-Cahn equation on variable grids. In order to preserve a modified energy dissipation law at the discrete level, a novel technique is designed to deal with the nonlinear term. The stability and convergence analysis of the WSBDF2 method are rigorously proved by the energy method under the adjacent time-step ratios $r_s \geq 4.8645$. Finally, numerical experiments are implemented to illustrate the theoretical results. The proposed approach is applicable for the Cahn-Hilliard equation.

Key words. Weighted and shifted two-step BDF method, variable step size, Allen-Cahn equation, stability and convergence analysis.

1. Introduction

The objective of this paper is to provide a rigorous stability and convergence analysis on variable grids for solving the Allen-Cahn equation [1]

$$\partial_t u - \varepsilon^2 \Delta u + f(u) = 0, \quad (x,t) \in \Omega \times (0,T] \tag{1}$$

with the initial condition $u(0) = u_0$ and periodic boundary conditions. Here, the nonlinear bulk force is defined as $f(u) = F'(u) = u^3 - u$, and the parameter $\varepsilon > 0$ denotes the interfacial width. The Allen-Cahn equation can be regarded as an L^2-gradient flow of the following free energy functional:

$$E[u] = \int_\Omega \left(\frac{\varepsilon^2}{2} |\nabla u|^2 + F(u) \right) dx \quad \text{with} \quad F(u) = \frac{1}{4}\left(u^2 - 1\right)^2. \tag{2}$$

In other words, the Allen-Cahn equation (1) satisfies the energy dissipation law:

$$\frac{dE[u]}{dt} = -\int_\Omega |\partial_t u|^2 \, dx \leq 0. \tag{3}$$

Let $N \in \mathbb{N}$ and consider the nonuniform time levels $0 = t_0 < t_1 < \cdots < t_N = T$ with the time-step $\tau_k = t_k - t_{k-1}$ for $1 \leq k \leq N$. For any time sequence $\{v^n\}_{n=0}^N$, we denote

$$\nabla_\tau v^n := v^n - v^{n-1}, \quad n \geq 1.$$

For $k = 1, 2$, let $\Pi_{n,k} v$ denote the interpolating polynomial of a function v over $k+1$ nodes $t_{n-k}, \cdots, t_{n-1}$ and t_n. Taking $v^n = v(t_n)$, and using the Lagrange interpolation, the one-step BDF formula yields $D_1 v^n := (\Pi_{n,1} v)'(t) = \nabla_\tau v^n / \tau_n$ for $n \geq 1$, and the two-step BDF formula reads

$$D_2 v^n = (\Pi_{n,2} v)'(t_n) = \frac{1 + 2r_n}{\tau_n(1 + r_n)} \nabla_\tau v^n - \frac{r_n^2}{\tau_n(1 + r_n)} \nabla_\tau v^{n-1}, \quad n \geq 2,$$

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*Corresponding author.

where the adjacent time-step ratios are defined by

$$r_1 := 0, \quad r_n := \frac{\tau_n}{\tau_{n-1}}, \quad n \geq 2.$$

Similarly, we construct the shifted two-step BDF formula as follows:

$$\widetilde{D}_2 v^n = (\Pi_{n,2} v)'(t_{n-1}) = \frac{1}{\tau_n(1+r_n)} \nabla_\tau v^n + \frac{r_n^2}{\tau_n(1+r_n)} \nabla_\tau v^{n-1}, \quad n \geq 2.$$

Thus, the weighted and shifted two-step BDF (WSBDF2) formula is defined by

$$\mathcal{D}_2 v^n := \theta D_2 v^n + (1-\theta)\,\widetilde{D}_2 v^n, \quad n \geq 2,$$

i.e.,

$$\mathcal{D}_2 v^n = \frac{1+2\theta r_n}{\tau_n(1+r_n)} \nabla_\tau v^n + \frac{(1-2\theta) r_n^2}{\tau_n(1+r_n)} \nabla_\tau v^{n-1}, \quad \theta \in [1/2, 1]\,. \tag{4}$$

It is noted that the WSBDF2 formula (4) is a flexible form between the Crank-Nicolson ($\theta = 1/2$) and the two-step BDF ($\theta = 1$) approximations for the discretization of the first order derivative $\partial_t u$ on nonuniform meshes. The WSBDF2 formula (4) coincides with the implicit-explicit multistep method of order 2 in [21]. However, the construction techniques are significantly different. In [21], the coefficients are derived from order conditions. In contrast, the WSBDF2 formula (4) is constructed by the weighted and shifted technique, which is simpler and more flexible to design the high-order methods, such as the weighted and shifted seven-step BDF method [3].

Since the WSBDF2 formula requires two starting values, we use the weighted and shifted one-step BDF formula to compute first-level solution u^1 by

$$\mathcal{D}_2 v^1 := \theta (\Pi_{1,1} v)'(t_1) + (1-\theta)\,(\Pi_{1,1} v)'(t_0) = \nabla_\tau v^1/\tau_1. \tag{5}$$

We recursively define a sequence of approximations u^n to the nodal values $u(t_n)$ of the exact solution by the WSBDF2 method,

$$\mathcal{D}_2 u^n - \varepsilon^2 \left(\theta \Delta u^n + (1-\theta)\Delta u^{n-1}\right) + H(u^n) = 0, \quad n \geq 1, \tag{6}$$

where the initial data is $u^0 = u_0$, and the nonlinear term $H(u^n)$ is constructed by

$$\begin{aligned} H(u^n) &= \theta f(u^n) + \big(1-\theta\big) f(u^{n-1}) - \frac{\theta}{2}\big(u^n + u^{n-1}\big)\big(u^n - u^{n-1}\big)^2 \\ &= \frac{\theta}{2}\left((u^n)^3 + (u^n)^2 u^{n-1} + u^n (u^{n-1})^2 - (u^{n-1})^3\right) \\ &\quad - \theta u^n + (1-\theta)\left((u^{n-1})^3 - u^{n-1}\right). \end{aligned} \tag{7}$$

The WSBDF2 formula (4) can be viewed as a discrete convolution summation:

$$\mathcal{D}_2 v^n = \sum_{k=1}^{n} b_{n-k}^{(n)} \nabla_\tau v^k, \quad n \geq 1, \tag{8}$$

where the discrete convolution kernels $b_{n-k}^{(n)}$ are defined by $b_0^{(1)} = 1/\tau_1$, and for $n \geq 2$,

$$b_0^{(n)} := \frac{1+2\theta r_n}{\tau_n(1+r_n)}, \quad b_1^{(n)} := \frac{(1-2\theta) r_n^2}{\tau_n(1+r_n)} \quad \text{and} \quad b_j^{(n)} := 0, \quad 2 \leq j \leq n-1. \tag{9}$$