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EFFICIENT AND LONG-TIME ACCURATE SECOND-ORDER DECOUPLED METHOD FOR THE BLOOD SOLUTE DYNAMICS MODEL

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Abstract. In this paper, we study the blood solute dynamics model to understand the relationship between the widespread pathologies of the vascular system, the specific features of the blood flow in a diseased district, and the effect of the flow pattern on the transfer processes of solute within arterial lumen and wall. The proposed finite element algorithm is based on the second-order backward differentiation formula and the explicit treatment of the coupling terms, which allow us to solve the decoupled Navier-Stokes equations, advection-diffusion equation, and pure diffusion equation at each time step. We derive the unconditional and long-time stability in the sense that the solution remains uniformly bounded in time, leading to uniform time error estimation. The long-time accurate behavior is one of the most desirable physical processes for the development of cardiovascular diseases that occurs over long-time scale. To validate the proposed method and demonstrate the exclusive features of the blood solute dynamical model, we perform four numerical experiments. Moreover, the impact of the development of atherosclerosis lesion and abdominal aortic aneurysm are studied by illustrating the complicated flow characteristics, streamlines, pressure contours, solute concentration, wall shear stress, and long-time accuracy on the several geometrical setups for the physiological interests.

Key words. Blood solute dynamics, second-order method, partitioned algorithm, unconditional stability, long-time stability.

1. Introduction

Over the past few decades, atherosclerosis and aneurysm considered as the most prevalent kind of cardiovascular diseases, have been studied extensively to identify the causes, genesis and the risk factors to achieve some methodologies for improving the human health by developing new prophylactic, diagnostic and therapeutic procedures [1, 3, 5, 4, 2, 6, 7, 8, 9]. Atherosclerosis is the hardening of large arteries due to the penetration and the development of the fatty plaque within the arterial wall, which leads to a gradual narrowing of the arteries. On the other hand, abdominal aortic aneurysms (AAAs) occur in the abdominal aortic artery where the artery has a balloon-shaped expansion; hence, the increase in the lumen diameter reaches up to 50 % of its standard diameter [2, 10, 11, 12, 13, 14]. Thus, the study of such cardiovascular diseases (atherosclerosis and aneurysm) involves the contribution of mass transport across the permeable endothelial layer and fluid dynamics of blood. On the other hand, in [10, 15], authors identified that hypertension is one of the main reason which enhanced the inner arterial wall due to the significant effects on the macromolecule distribution. Moreover, the dependence of shear stress on the solute transport from blood to the stenosis artery wall has been discussed in [11, 18, 16, 17].

Based on the arterial anatomy and the high dependence of the development of lesions on the transport procedure within the arteries, scientists have been set up

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some partial differential equation models and numerical tools to study the blood flow. In [16, 17, 19], the authors introduced the *wall-free* model to describe the dynamic of the macromolecules (Low-Density-Lipoprotein (LDL), albumin, oxygen, etc.) from the lumen to the arterial wall through the endothelial layer, which is considered as the common interface. The *wall-free* model is based on the coupling of the Navier-Stokes equations to describe the blood motion in the vascular tissue, the solute concentration in the lumen is modeled by the advection-diffusion equation, while the solute dynamics inside the wall is considered as a given quantity. On the other hand, a more sophisticated and realistic blood solute dynamics model is proposed in [20, 21, 22, 23] by considering the *fluid-wall* phenomena. In the *fluid-wall* model the solute dynamics into the vascular wall is taken into account and modeled by the pure diffusion equation by neglecting the convective field due to the small variety of blood velocity in the wall. Furthermore, the conservation of solute concentration and the exchange of flux of solute through the permeable membrane from lumen to the arterial wall are governed by two coupling conditions [22, 23]. In nature, the blood vessel is elastic, deform due to the cardiovascular system, while certain studies assumed the arterial wall as a rigid structure [19, 24].

It is worthwhile noting that many numerical methods have been developed to decouple the original problem for the accurate and efficient resolution of the multi-domain, and multiphysics problems [25, 26, 29, 27, 28, 33, 30, 31, 32]. The long-time feature is essential for the realization of the development of cardiovascular diseases, thus the long-time accuracy of the algorithm is highly desirable. Jiang investigated a second-order ensemble method based on a blended backward differentiation time-stepping algorithm for the time-dependent Navier-Stokes equations in [34]. The unconditional long-time stability for a particular velocity-vorticity discretization of the 2D Navier-Stokes equations studied in [35]. Hou et al. derived the second-order convergence of a projection scheme for the incompressible Navier-Stokes equations in [36]. Besides, the second-order schemes have been considered to decouple the system of Navier/Stokes-Darcy equations extensively. In [37], Layton et al. proposed uncoupled Crank-Nicolson Leapfrog ($CNLF$) and $BD2$-$AB2$ schemes and derived the stability of the system. Chen et al. [38] proved the unconditional and uniform stability of two second-order $BDF2$ and $AMB2$ schemes, which imply the uniform control of the error. In [39], a third-order scheme has been studied, while in [40], the authors presented a second-order decouple scheme and uncouple the velocity and pressure by artificial compression method. A second-order partitioned method with different subdomain time steps for the evolutionary Stokes-Darcy system presented in [41]. In [42], Heister et al. considered the decoupled, unconditionally stable, higher-order discretizations for MHD flow simulation. The long-time stability of the extrapolated $BDF2$ time-stepping methods for the Navier-Stokes equations and related multiphysics problems have been studied in [43]. Ravindran [44], proposed the second-order $BDF2$ partitioned time-stepping algorithm for solving the transient viscoelastic fluid flow. Moreover, some numerical methods for the blood solute dynamics model have been presented. In [47], the author proposed a robust modified characteristics variational multiscale (MCVMS) method, which is based on the combination of the characteristics temporal discretization to deal with the difficulty caused due to the nonlinear terms, and the projection-based variational multiscale (VMS) technique to stabilize the spurious oscillation caused by the lower diffusivity of the solute concentration. Also, an IMEX scheme and a data-passing scheme for the blood model were considered in [46].