INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 23, Number 2, Pages 163–183

doi: 10.4208/ijnam2026-1008

AN ASYMPTOTIC EXPANSION BASED DEEP NEURAL NETWORKS FOR SOLVING SINGULARLY PERTURBED PROBLEMS WITH EXPONENTIAL BOUNDARY LAYERS

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Abstract. Physics-informed neural networks (PINNs) have emerged as a powerful framework for solving partial differential equations (PDEs). However, their effectiveness deteriorates when applied to singularly perturbed problems, where solutions exhibit steep gradients and boundary layers confined to narrow regions features that standard PINNs architectures often fail to capture. To overcome these obstacles, we propose a novel, mesh-free deep neural networks (DNNs) method that incorporates asymptotic information through a systematic decomposition of the solution into smooth and sharp components. By capitalizing on the ability of DNNs to excel at approximating smooth functions, our approach constructs a series of moderately sized networks tailored to different elements of the solution. This strategy enables uniform approximation accuracy across a wide range of perturbation parameters, maintaining robustness and efficiency even in the extreme regime where the perturbation parameter is as small as 10^{-16}, near machine precision. Key advantages of the proposed method include its conceptual simplicity, full independence from mesh requirements, and ease of implementation. Extensive numerical experiments confirm that our approach delivers significantly improved accuracy and efficiency compared to standard PINNs, demonstrating its potential as a robust and versatile framework for tackling singularly perturbed problems.

Key words. Singular perturbed problem, Uniform convergence, Deep neural networks, Asymptotic expansion.

1. Introduction

Singularly perturbed problems arise frequently in a broad spectrum of disciplines, including fluid dynamics, optical control, financial mathematics, biology, chemistry, and engineering sciences. These problems are characterized by the presence of a small parameter (e.g., ϵ in (2)) within the governing equations, leading to solutions that exhibit thin boundary layers with rapid spatial variations. The presence of such layers poses substantial challenges to the design of effective numerical methods. Classical numerical schemes often fail to resolve these layers accurately when the perturbation parameter becomes sufficiently small, as they typically lack uniform convergence with respect to the parameter. As a result, the development of numerical methods that guarantee uniform convergence remains a fundamental and active task. Two widely adopted strategies for addressing these challenges are the fitted operator method and the fitted mesh method, both of which have inspired the development of a variety of uniformly convergent numerical schemes over the past decades; see, for instance, [2, 15, 26, 29, 28, 13, 30, 25].

Motivated by the limitations of traditional methods, recent years have witnessed growing interest in machine learning-based approaches–particularly deep neural networks (DNNs)–for solving PDEs. DNNs have achieved remarkable success in scientific computing, largely due to their universal approximation property. Unlike traditional numerical methods such as finite difference, finite element, spectral,

Received by the editors on July 2, 2025 and, accepted on September 1, 2025.
2000 *Mathematics Subject Classification.* 68T07, 65L11, 65C05.
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and finite volume methods, DNNs offer significant advantages when solving high-dimensional problems that suffer from the "curse of dimensionality", referring to the exponential growth in computational complexity as the number of dimensions increases [12]. Furthermore, DNNs have demonstrated notable capabilities in addressing low-dimensional problems with low regularity or sharp solutions. Widely adopted methods for solving PDEs with DNNs include physics-informed neural networks (PINNs), which minimize least-squares or variational loss functions [4, 19, 14]; the deep Ritz method (DRM), which employs energy functionals as loss functions [8, 27]; and the operator learning method, which seeks to learn solution operators directly [18], among others.

However, despite these successes, directly applying DNNs-based methods such as PINNs to singularly perturbed problems remains highly nontrivial. It is well-documented that neural networks struggle to accurately capture sharp variations in functions [23], making them ill-suited for resolving the thin boundary layers characteristic of such problems. Consequently, naive implementations of PINNs often suffer from large generalization errors in the singularly perturbed regime. To overcome these challenges, several enhancements have been proposed to improve the performance of PINNs on problems with sharp solution features. For instance, one common strategy is to increase the density of residual points in regions exhibiting rapid variation, thereby providing more training information where it is most needed [21, 22]. Another effective approach is adaptive sampling, which dynamically adjusts the distribution of training points based on error indicators or gradient magnitudes, so that computational effort is concentrated on regions with steep solution gradients [9, 20]. In addition, more tailored DNN-based approaches have been developed specifically for singular perturbation problems, including a boundary-layer PINNs based on singular perturbation theory [3], and an asymptotic parameter PINNs (PAPINNs), which first trains the network with large perturbation parameters to learn smooth solutions and then refines the network to resolve sharp features [5]. Other strategies combine deep operator networks with Shishkin mesh points for loss evaluation [7], or employ Galerkin neural network procedures to accurately capture boundary layer behaviors in stress functions [1].

In addition to these algorithmic improvements, a number of recent studies have systematically compared various PINNs-based methods for problems with sharp gradients and have introduced novel architectures that embed asymptotic knowledge directly into the learning process. For example, [6] presents a comparative study of several representative approaches, including Classic-PINNs, Deep-TFC, PIELM, and X-TFC. A novel semi-analytic SL-PINNs framework is proposed in [11], where explicit correctors capturing the singular behavior of stiff boundary-layer solutions are either embedded into the PINNs architecture or jointly trained with the network. The Component Fourier Neural Operator (ComFNO), an advanced operator learning method built upon the Fourier Neural Operator (FNO), further incorporates prior knowledge from asymptotic analysis to improve accuracy and generalization [17]. In addition, [10] develops a hybrid numerical approach for the singularly perturbed Robin parabolic convection-diffusion problem, combining a Shishkin mesh with central difference discretization in the inner region, an upwind midpoint scheme in the outer region, and the implicit Euler method for temporal discretization.

While the aforementioned methods have demonstrated success in various settings, they still face challenges in accurately capturing boundary layer behavior when the perturbation parameter becomes extremely small (e.g., $\epsilon = 10^{-16}$). In