INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 23, Number 2, Pages 184–222

doi: 10.4208/ijnam2026-1009

ANALYSIS OF ANY ORDER RUNGE-KUTTA SPECTRAL VOLUME SCHEMES FOR 1D HYPERBOLIC EQUATIONS

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Abstract. In this paper, we analyze any-order Runge-Kutta spectral volume schemes (RKSV(s,k)) for solving the one-dimensional scalar hyperbolic equation. The RKSV(s,k) was constructed by using the s-th explicit Runge-Kutta method in time-discretization which has *strong-stability-preserving* (SSP) property, and by letting a piecewise $k-$th degree($k \geq 1$ is an arbitrary integer) polynomial satisfy the local conservation law in each control volume designed by subdividing the underlying mesh with k Gauss-Legendre points (LSV) or right-Radau points (RRSV). For the RKSV(s,k), we would like to establish a general framework which use the matrix transferring process technique for analyzing the stability and the convergence property. The framework for stability is evolved based on the energy equation, while the framework for error estimate is evolved based on the error equation. And the evolution process is represented by matrices. After the evolution is completed, three key indicative pieces of information are obtained: the termination factor ζ, the indicator factor ρ, and the final evolved matrix. We prove that for the RKSV(s,k), the *stability* holds and the L_2 norm error estimate is $\mathcal{O}(h^{k+1} + \tau^s)$, provided that the CFL condition is satisfied. Our theoretical findings have been justified by several numerical experiments.

Key words. Spectral volume (SV) methods, L_2-norm stability, error estimates, any-order Runge-Kutta method.

1. Introduction

In this paper, we propose an analysis framework to obtain the L_2-norm stability and the convergence properties of the explicit any order Runge-Kutta spectral volume schemes(RKSV(s,k)), when solving the hyperbolic equation

$$\partial_t u + \partial_x u = 0, \quad (x,t) \in [a,b] \times (0,T]. \tag{1}$$

The equation with initial condition

$$u(x,0) = u_0(x), \quad x \in [a,b], \tag{2}$$

where u_0 is a known function, and with the homogeneous boundary condition $u(a,t) = 0, t \in [0,T]$ or with the periodic boundary condition $u(a,t) = u(b,t), t \in [0,T]$.

For temporal discretization, we choose the Runge-Kutta method. As is well known, there are many ways to select coefficients for constructing the discrete-time derivatives of an s-th order Runge-Kutta method. Different choices of Runge-Kutta methods for different parameters exhibit different performances, with some able to maintain *total-variation-diminishing*(TVD) properties[1], some able to maintain *strong-stability-preserving* (SSP) properties [2] and so on. These properties are highly applicable to the numerical construction of conservation equations. In this paper, we consider the type of time-marching has been later termed SSP, which is widely applied in the analysis of nonlinear stability including the TVD property and the positivity-preserving property [3] for nonlinear conservation laws. This

Received by the editors on September 19, 2024 and, accepted on August 25, 2025.
2000 *Mathematics Subject Classification.* 35L65, 65M08, 65M12.
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class of methods has a unified formula for deriving and calculating the coefficients, enabling us to establish a unified framework for the analysis of any order.

For spatial discretization, we employ the spectral volume(SV) method, which was first proposed in 2002 by Wang [4] for solving hyperbolic equations. The basic idea of the SV method is to ensure that a (discontinuous) piecewise high-order polynomial satisfies sub-element-level conservation laws. Due to its numerous advantages, such as local conservation properties, geometric and mesh flexibility, high-order accuracy in smooth regions, and high resolution in discontinuous areas [5], the SV method has been successfully applied to various partial differential equations (PDEs). Applications include Burgers' equation [6], the shallow water equation [7, 8], the Euler equation [9, 10, 11, 12, 13], electromagnetic field equations [14], and the Navier-Stokes equation [15, 16, 17].

In [18], Wang and Shu classify the spectral volume (SV) method as a special Petrov-Galerkin method. While widely applied, the mathematical theory of the SV method, particularly its stability and convergence properties, has been less developed. Early theoretical studies focused on the stability of low-order schemes over uniform grids. Abeele et al. discovered that subdividing points significantly influence stability and noted that third and fourth-order SV schemes based on Gauss-Lobatto points are weakly unstable [19, 20]. Zhang and Shu proved the stability of 1st, 2nd, and 3rd order SV schemes on 1D uniform meshes using Fourier analysis [18]. Recently, Cao and Zou [21] developed a uniform framework for analyzing the stability, convergence, and superconvergence of SV schemes over nonuniform meshes for 1D scalar hyperbolic equations, extending their analysis to 1D and 2D hyperbolic equations with degenerate variable coefficients [22, 23]. Their main approach involves constructing a novel trial-to-test space mapping, allowing the SV method to be reformulated as a special Galerkin method.

In [24], Wei et al. analyzed the stability and convergence properties of two fully discrete schemes: the forward Euler spectral volume scheme and the second-order Runge-Kutta spectral volume scheme. They utilized a unified formula to derive and calculate the coefficients, demonstrating that under various Courant-Friedrichs-Lewy (CFL) conditions, both schemes exhibit optimal convergence rates. The analysis involves checking the eigenvalues of the amplification matrix and ensuring they lie within the unit circle for stability. This paper builds on these findings by examining the stability and convergence properties of any-order Runge-Kutta spectral volume (RKSV) methods for hyperbolic equations, further extending the theoretical framework and providing a comprehensive analysis of stability criteria and convergence orders for these fully discrete schemes.

For the fully discrete RKSV for linear coefficient hyperbolic equations, we aim to establish a general framework for analyzing stability and optimal order convergence properties. It is worth noting that the SV method is essentially a Petrov-Galerkin method rather than a pure Galerkin one [18]. In recent years, there has been extensive literature on the properties of fully discrete Runge-Kutta discontinuous Galerkin methods (RKDG) schemes, analyzing the stability and optimal order convergence properties for the explicit third-order and fourth-order RKDG methods [25, 26, 27, 28]. In [29], Xu et al. propose a unified framework to investigate the L_2-norm stability of the explicit RKDG using the matrix transferring process technique. The main approach in the stability analysis is to establish a robust energy equation that clearly reflects the evolution of the L_2-norm of the numerical solution and explicitly shows the stability mechanism hidden in the fully discrete scheme.