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ON *A PRIORI* ERROR ANALYSIS OF DISCONTINUOUS GALERKIN METHOD FOR THE VLASOV-NONSTATIONARY STOKES SYSTEM

KRISHAN KUMAR, HARSHA HUTRIDURGA, AND AMIYA K. PANI

Abstract. In the first part of this paper, the uniqueness of a strong solution is established for the Vlasov-unsteady Stokes problem in 3D. The second part deals with a semi discrete scheme, which is derived as a result of spatial discretization of the coupled system of Vlasov and Stokes equations for the 2D problem by discontinuous Galerkin methods, while keeping temporal variable continuous. The proposed semi-discrete scheme preserves both mass and momentum conservation properties. Based on the orthogonal L^2 as well as the Stokes projections, error estimates in the case of smooth compactly supported initial data are derived by employing a variant of nonlinear Grönwall's lemma in a crucial way. Moreover, the generalization of error estimates to 3D problem is also briefly discussed. Finally, using a time splitting algorithm as the phase space is four dimensional, some numerical experiments are conducted, whose results confirm our theoretical findings.

Key words. Vlasov-nonstationary Stokes, uniqueness in 3D, discontinuous Galerkin method, conservation properties, error estimates, nonlinear version of Grönwall's lemma, time-splitting scheme and numerical experiments.

1. Introduction

This paper develops and analyses a discontinuous Galerkin method for the following coupled system of Vlasov-nonstationary Stokes equations:

$$\begin{cases} \partial_t f + v\cdot\nabla_x f + \nabla_v\cdot\Big((\boldsymbol{u}-v)\,f\Big) = 0 & \text{in } (0,T)\times\Omega_x\times\mathbb{R}^d, \\ f(0,x,v) = f_0(x,v) & \text{in } \Omega_x\times\mathbb{R}^d. \end{cases} \tag{1}$$

$$\begin{cases} \partial_t\boldsymbol{u} - \Delta_x\boldsymbol{u} + \nabla_x p = \displaystyle\int_{\mathbb{R}^d}(v-\boldsymbol{u})\,f\,\mathrm{d}v & \text{in } (0,T)\times\Omega_x, \\ \nabla_x\cdot\boldsymbol{u} = 0 & \text{in } \Omega_x, \\ \boldsymbol{u}(0,x) = \boldsymbol{u_0}(x) & \text{in } \Omega_x. \end{cases} \tag{2}$$

Here, Ω_x is a d-dimensional torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ for $d = 2, 3$, $\boldsymbol{u}(t,x)$ denotes the fluid velocity, $p(t,x)$ is the fluid pressure, and $f(t,x,v)$ represents the droplet distribution function. In this paper, we take $\mathbb{T}^d = [0,M]^d, M > 0$ for $d = 2, 3$, with periodic boundary conditions. This system arises in the mathematical model of the thin sprays of droplets in a gas medium, which is dispersed into a background fluid medium. The dispersed phase is modelled by a kinetic equation called Vlasov equation, where as, the background fluid medium is described by the nonstationary Stokes equation. The coupling in Vlasov part occurs via a drag force and in Stokes system by a Brinkman force which acts as a forcing term. Study of this type of models comprising of a kinetic equation for the dispersed phase and a fluid equation for the gas goes back to the works of O'Rourke [26] and Williams [32] (also, see [10]).

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Earlier, Hamdache [19] has discussed the global existence of weak solutions to the system (1)-(2) with homogeneous Dirichlet boundary condition for the fluid velocity and with specular reflection boundary condition for the droplet distribution. Various kinetic fluid equations have been discussed in the literature, say, for example Vlasov-Burgers equations [16, 17]; Vlasov-Euler equations [4]; Vlasov-Navier-Stokes equations [5, 9, 34, 22] and references, therein.

This paper deals with some qualitative and quantitative properties of solutions to our system (1)-(2). Note that a proof of the global-in-time existence of strong solution to the system (1)-(2) in 3D is established in [9], but uniqueness is missing there, therefore, our first attempt is to prove uniqueness result in 3D.

Since our major emphasis in the subsequent part of this paper is on developing and analysing an appropriate numerical scheme for the system (1)-(2), which is motivated by [12, 20, 13, 14] for the Vlasov-Poisson system, [25] for the Vlasov-steady Stokes and [7, 33] for the Vlasov-Maxwell system, we apply discontinous Galerkin methods for both kinetic and nonstationary Stokes equations to discretize in both x and v variables keeping time variable continuous. Thereby, we derive a semi-discrete scheme. Note that the kinetic equation is a transport problem which conserves total mass. As discontinuous Galerkin(dG) methods have the property to preserve the conservation property, dG turns out to be a method of choice for the kinetic equation. For the time dependent Stokes system, we apply dG, but other methods like mixed finite element method, the local discontinuous Galerkin or any conforming or nonconforming numerical method can also be employed. Our main contributions in this paper are as follows:

- Uniqueness of strong solution in three dimensions is proved for the Vlasov-unsteady Stokes system. Earlier in [24], there is an error in the proof of uniqueness and a correct proof of uniqueness is given in Section 2 of this paper. This can be achieved after proving some higher moment estimates for the droplet distribution and regularity results for the fluid velocity vector.
- DG methods are proposed for numerical approximations and are shown to conserve mass and the total momentum, which confirm similar conservation properties for the continuous system.
- Error estimates are derived for the fluid velocity and the fluid pressure approximations using Stokes projection in 2D setup, while in 3D case some remarks are given in the subsection 4.4. Moreover, error estimate for the approximation of the droplet distribution is obtained using the orthogonal L^2-projection. The error analysis uses a variant of the nonlinear Grönwall's inequality in a crucial way.
- Since phase space is four or six dimensions depending on $d = 2$ or $d = 3$, a Lie splitting in time marching combined with dimension splitting in the phase space is proposed for computational experiments, whose numerical results confirm our theoretically findings.

This paper is organized as follows. In Section 2, we deal with certain qualitative and quantitative properties of the solutions to the continuum model. We further prove uniqueness result for strong solutions to (1)-(2). Section 3 introduces a semi-discrete dG-dG numerical scheme and analyses some of its properties. *A priori* error estimates for the semi-discrete method are established in Section 4. Further, some comments on 3D problems are presented. Section 5 deals with some numerical experiments based on a time-splitting schemes combined with dG methods, whose results confirm our theoretical findings. Finally in Section 6, some concluding remarks are given.