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A Piecewise Modified Matrix Padé-type Approximation of Hybrid Order in the Interval [0, 1]

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Abstract. In this paper, we introduce a piecewise modified matrix Padé-type approximation of hybrid order in the interval [0, 1]. It yields highly accurate results and exact values at some given points. The accuracy of this approximation increases as the order or the node increases. This method can be applied to approximate the exponential function. The explicit formula for computing the matrix exponential is presented. A numerical example is given to illustrate the effectiveness of this method.

Keywords: matrix Padé-type approximation, piecewise interpolation, matrix exponential.

1. Introduction

Padé-type approximation has been widely used in various fields of mathematics, physics, and engineering since it was first introduced by Brezinski [1, 2]. In the last thirty years, it has received a lot of attention and has been generalized to many cases by some authors, for example, Arioka [3], Daras [4], Salam [8], Thukral [9], Gu and Shen [6].

Following the idea of scalar Padé-type approximation [2], Gu [5] gave a matrix Padé-type approximation whose denominator is a scalar polynomial by means of a matrix-valued linear functional on the polynomial space. The matrix Padé-type approximation [5, 7] is a good approximation in a region near the origin, but may not be accurate at other points. To improve the accuracy of such approximation, we develop an interpolation technique for generating a new rational approximation with high accuracy.

In this paper, matrix Padé-type approximation is modified by an interpolation polynomial. Based on the modified matrix Padé-type approximation, we construct a piecewise modified matrix Padé-type approximation of hybrid order in the interval [0, 1]. This new rational approximation to a matrix function gives more accurate results and exact values at certain selected points in the interval [0, 1]. In addition, we use this new method to approximate the matrix exponential. The practical formula for computing the matrix exponential is presented. It is shown by a numerical example of the matrix exponential that the accuracy of the approximation increases as the order or the node increases.

2. Main Results

Let $F(t)$ be a given power series at t_k with $s\times s$ matrix coefficients

$$F(t)=\sum_{i=0}^{\infty}C_i^{(t_k)}(t-t_k)^i\,,\quad C_i^{(t_k)}\in C^{s\times s}\,,\quad C_0^{(t_k)}\neq 0\,,\quad k\in \mathrm{N}. \tag{1}$$

Denote P the set of scalar polynomials in one real variable whose coefficients belong to the complex field C. Let $\phi^{(l)}:\mathrm{P}\rightarrow C^{s\times s}$ be a matrix-valued linear functional on P, acting on $(x-t_k)$, defined by

$$\phi^{(l)}((x-t_k)^i)=C_{l+i}^{(t_k)}\,,\quad i,l=0,1,\cdots,$$

where

$$\phi^{(0)}((x-t_k)^i)=\phi((x-t_k)^i)=C_i^{(t_k)}\,,\quad i=0,1,\cdots, \tag{2}$$

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with $C_i^{(t_k)} = 0$ for $i < 0$.

For the given power series (1), it follows from (2) that

$$\begin{aligned}&\phi((1-(x-t_k)(t-t_k))^{-1})\\ &=\phi(1+(x-t_k)(t-t_k)+(x-t_k)^2(t-t_k)^2+\cdots)\\ &=C_0^{(t_k)}+C_1^{(t_k)}(t-t_k)+C_2^{(t_k)}(t-t_k)^2+\cdots\\ &=F(t).\end{aligned} \tag{3}$$

Let v be an arbitrary polynomial of degree n

$$v(t)=b_0+b_1(t-t_k)+\cdots+b_n(t-t_k)^n, \quad b_n \neq 0, \tag{4}$$

and define the matrix polynomial W_{mn} by

$$W_{mn}(t)=\phi\left(\frac{(x-t_k)^{m-n+1}v(x)-(t-t_k)^{m-n+1}v(t)}{x-t}\right). \tag{5}$$

Note that ϕ acts on $(x-t_k)$ and W_{mn} is a matrix polynomial of degree m . Define

$$\begin{aligned}q_{mn}^{t_k}(t)&=(t-t_k)^n v((t-t_k)^{-1}+t_k),\\ P_{mn}^{t_k}(t)&=(t-t_k)^m W_{mn}((t-t_k)^{-1}+t_k).\end{aligned} \tag{6}$$

Theorem 1. If $q_{mn}^{t_k}(t_k) \neq 0,$ then

$$F(t)-\frac{P_{mn}^{t_k}(t)}{q_{mn}^{t_k}(t)}=O((t-t_k)^{m+1}), \quad t \to t_k.$$

Proof. Note that ϕ is a matrix-valued linear functional on P, acting on $(x-t_k)$. From (3), (5) and (6), we get

$$\begin{aligned}P_{mn}^{t_k}(t)&=(t-t_k)^m W_{mn}((t-t_k)^{-1}+t_k)\\ &=(t-t_k)^m\phi\left(\frac{(t-t_k)^{-(m-n+1)}v((t-t_k)^{-1}+t_k)-(x-t_k)^{m-n+1}v(x)}{((t-t_k)^{-1}+t_k)-x}\right)\\ &=\phi\left(\frac{(t-t_k)^n v((t-t_k)^{-1}+t_k)-(t-t_k)^{m+1}(x-t_k)^{m-n+1}v(x)}{1-(x-t_k)(t-t_k)}\right)\\ &=q_{mn}^{t_k}(t)F(t)-(t-t_k)^{m+1}\phi^{(m-n+1)}\left(\frac{v(x)}{1-(x-t_k)(t-t_k)}\right).\end{aligned}$$

Thus, the result follows if $q_{mn}^{t_k}(t_k) \neq 0.$

Definition 1. Let $R_{mn}^{t_k}(t)=P_{mn}^{t_k}(t)/q_{mn}^{t_k}(t).$ Then $R_{mn}^{t_k}(t)$ is called a matrix Padé-type approximation of order (m,n) for $F(t)$ at $t=t_k$ and is denoted by $(m/n)_F^{t_k}(t)$, where $v(t)$ in (4) is called a generating polynomial of $(m/n)_F^{t_k}(t)$.

Note that the definition of matrix Padé-type approximation given in [5] is a special case of Definition 1 here for $t_k=0.$ We can also denote $(m/n)_F^0(t)$ simply by $(m/n)_F(t).$

Definition 2. Let $F(t)$ defined on the interval $[t_0,t_1]\subset[0,1]$ be given the values at $t=t_0$ and $t=t_1 (t_0 \neq t_1).$ Then a modified matrix Padé-type approximation (MMPTA) for $F(t)$ in the interval $[t_0,t_1]$ is defined by

$$(m/n)_F^{[t_0,t_1]}(t)=(m/n)_F^{t_0}(t)+\varepsilon_{mn}^{t_0}(t-t_0)^{m+1}, \tag{7}$$