ISSN 1746-7659, England, UK
Journal of Information and Computing Science
Vol. 6, No. 1, 2011, *pp*. 035-042

The Least Squares Solutions of Bisymmetric Matrix for Inverse Quadratic Eigenvalue Problem

Xiangrong Wang+, Chenggang Chen, Shimin Wan, Yandong Yuan

Department of Fundamental Subject, Tianjin Institute of Urban Construction, Tianjin , 300384, China

(*Received January 1, 2010, accepted March 22, 2010*)

Abstract. The inverse eigenvalue problem of constructing bisymmetric matrices M , C and K of size n for the quadratic pencil $Q(\Lambda) = MX\Lambda^2 + CX\Lambda + KX$ so that has a prescribed subset of eigenvalues and eigenvectors is discussed. A general expression of solution to the problem is provided. The set of such solutions is denoted by S_L . The optimal approximation problem associated with S_L is posed, that is: to find the nearest triple matrix $[\hat{M}, \hat{C}, \hat{K}]$ from S_L . The existence and uniqueness of the optimal approximation problem is discussed and the expression is provided for the nearest triple matrix.
Keywords: bisymmetric matrix, matrix equation, quadratic eigenvalue, inverse problem, SVD.

1. Introduction

Let $R^{n\times n}$ denote the set of $n\times n$ real matrices. $SR^{n\times n}$ denote the set of $n\times n$ real symmetric matrices. $ASR^{n\times n}$ be the set of $n\times n$ real anti-symmetric matrices, R^n denote the set of n dimensional vector. A^T is the transpose of matrix A . I_n is $n\times n$ unit matrix, $\|\bullet\|$ is Frobenius norm, $\|\bullet\|_2$ is 2-norm. e_i be i-th row of the unit matrix I_n . Let A be a real $m\times n$ matrix and let B be real $p\times q$ matrix. Then the Kronecker product of matrices A and B is defined as

$$A \otimes B := \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}. \tag{1}$$

That is, $A\otimes B$ is the $mp\times nq$ matrix formed all possible pairwise element products of A and B. If we let $vec(X)\in R^{mn}$ be the vector formed by the columns of a permutation matrix $X\in R^{m\times n}$.

Definition[1]**.** Let $A=(a_{ij})_{n\times n}$, $a_1=(a_{11},a_{21},\cdots,a_{n1})$, $a_2=(a_{22},a_{32},\cdots,a_{n2})$, $\cdots$,

$a_{n-1}=(a_{(n-1)(n-1)},a_{n(n-1)})$, $a_n=(a_{nn})$. Then we denote $vec_S(A)$ as follow

$$vec_S(A):=(a_1,a_2,\cdots,a_{n-1},a_n)^T \in R^{\frac{n(n+1)}{2}} . \tag{2}$$

Definition[2]**.** $A=(a_{ij})\in R^{n\times n}$ is termed bisymmetric matrix, if

$$a_{ij}=a_{ji}=a_{n-j+1,n-i+1},\ i,j=1,2,\cdots,n \tag{3}$$

Let $G=\{[X,Y,Z]\mid X\in BSR^{n\times n}, Y\in BSR^{n\times n}, Z\in BSR^{n\times n}\}$.

In this paper, we discuss the following problems:

Problem I. Given matrices $X\in R^{n\times p}$, $\Lambda\in R^{p\times p}$, find $[\tilde{M},\tilde{C},\tilde{K}]\in G$ such that

+ Corresponding author. Tel.: +86-022-23085200.
E-mail address: ydyuan196@sina.com.

$$\left\|\tilde{M}X\Lambda^2+\tilde{C}X\Lambda+\tilde{K}X\right\|=\min_{[M,C,K]\in G}\left\|MX\Lambda^2+CX\Lambda+KX\right\| \tag{4}$$

where $\|\bullet\|$ is Frobenius norm.

Let $\tilde{G}=\{[M,C,K]\,|\,\left\|MX\Lambda^2+CX\Lambda+KX\right\|=\min,[M,C,K]\in G\}$.

Problem II. Find $[\tilde{M},\tilde{C},\tilde{K}]\in\tilde{G}$, such that

$$\|\tilde{M}\|^2+\|\tilde{C}\|^2+\|\tilde{K}\|^2=\min_{[M,C,K]\in\tilde{G}}(\|M\|^2+\|C\|^2+\|K\|^2) \tag{5}$$

where $\|\bullet\|$ is Frobenius norm.

2. The Solution Problem I

Lemma 1[1]**.** Let $A\in R^{m\times n}, b\in R^n$, then the sufficiency and necessary condition of the solution exist for linear equation $Ax=b$ as follow

$$AA^+b=b. \tag{6}$$

the general solution for linear equation $Ax=b$ can write as follow

$$x=A^+b+(I-A^+A)\tau, \tag{7}$$

where $\tau\in R^n$.

Lemma 2[1]**.** Let $A\in R^{m\times n}, b\in R^n$, then the least squares solution of incompatibility linear equation $Ax=b$ can write as follow

$$x=A^+b+(I-A^+A)\tau, \tag{8}$$

where $\tau\in R^n$.

For any k of positive integer, let

$$D_{2k}=\frac{1}{\sqrt{2}}\begin{pmatrix}I_k & S_k\\ S_k & -I_k\end{pmatrix},\ D_{2k+1}=\frac{1}{\sqrt{2}}\begin{pmatrix}I_k & O & S_k\\ O & \sqrt{2} & O\\ S_k & O & -I_k\end{pmatrix}, \tag{9}$$

where $S_k=(e_k,e_{k-1},\cdots,e_2,e_1)$.

We easy know, for any positive integer n, have $D_n^TD_n=I_n$, $D_n^T=D_n$, then D_n is symmetric orthogonal matrix.

Lemma 3[2]**.** For any n is odd number or even number, the sufficiency and necessary condition of $n\times n$ real matrix being bisymmetric matrix is

$$X=D_n\begin{pmatrix}X_1 & O\\ O & X_2\end{pmatrix}D_n, \tag{10}$$

where $X_1\in SR^{(n-k)\times(n-k)}$, $X_2\in SR^{k\times k}$, $k=[\frac{n}{2}]$, D_n as (9).

Lemma 4[2]**.** Given matrix $X\in R^{n\times n}$, then the sufficiency and necessary condition for $X\in SR^{n\times n}$ as follow

$$vec(X)=\Gamma_n vec_S(X) \tag{11}$$