ISSN 1746-7659, England, UK
Journal of Information and Computing Science
Vol. 6, No. 1, 2011, pp. 043-048

On the Positive and Negative Solutions of p -Laplacian BVP with Neumann Boundary Conditions

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(*Received September 24, 2009, accepted October 22, 2009*)

Abstract. In this paper, we consider the following Neumann boundary value problem

$$\begin{cases} \varphi_p(u'(x))' = |u(x)|^p u(x) \quad \lambda, \quad x \in (0,1), \\ u'(0) = 0 = u'(1) \end{cases}$$

Where $\lambda \in \mathbb{R}$ and p>1 are parameters. We study the positive and negative solutions of this problem with respect to a parameter r (i.e. $u(0) = r$) in all $\mathbb{R}^*$. By using a quadrature method, we obtain our results. Also we provide some details about the solutions that are obtained.
AMS subject classification: 34B15.

Keywords and phrases: Positive and negative solutions; Interior critical points; Quadrature method; Neumann boundary condition; p-Laplacian problem.

1. Introduction

Consider the nonlinear two point boundary value problem

$$-\varphi_p(u'(x))' = |u(x)|^p u(x) - \lambda, x \in (0,1) \tag{1}$$

$$u'(0) = 0 = u'(1) \tag{2}$$

where $\lambda \in \mathbb{R}$ and p >1 are parameters and $\varphi_p(x) := |x|^{p-2}x$ for all $x \neq 0$ and $\varphi_p(0) = 0$ where $\varphi_p(u')'$ is the one dimensional p -Laplacian operator. We study the positive and negative solution of this problem with respect to a parameter r (that is the value of the solutions at zero, i.e. $u(0) = r$). Also by using a quadrature method, we obtain our results. In [9] problem (1) with Dirichlet boundary value conditions have been studied by Ramaswamy for the case Laplacian and in [1] the same problem with the same boundary value conditions have been extended by Addou to the general quasilinear case p -Laplacian with p >1, i.e. $-\varphi_p(u'(x))' = |u|^p u - \lambda$. In [2] and [7] for semipositon problems with p –Laplacian operator, existence and multiplicity results have been established with Neumann boundary value conditions and Dirichlet boundary value conditions, respectively. In [5], for semipositon and positon problems have been studied by Anuradha, Maya and Shivaji by using a quadrature method with Neumann-Robin boundary conditions and Laplacian operator. In [8] for semipositone problems, existence and multiplicity results have been established with Laplacian operator and Neumann boundary value conditions. Also, in [3] and [6] for semipositon problems with Laplacian operator have been studied for solution curves with Dirichlet boundary value conditions.

This paper is organized as follows. In Section 2, we first state some remarks and then our main results and finally in Section 3, we provide the proof of our main results that contains several lemmas.

2. Main Results

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By a solution of (1)- (2) we mean a function $u \in C^1([0,1])$ for which $\varphi_p(u'(x)) \in C^1([0,1])$ and both the equation and the boundary value conditions are satisfied.

Remark 1 If u is a solution to (1)- (2) at λ then $-u$ is a solution to (1)- (2) at $-\lambda$.

Remark 2 Let u is a solution to (1)- (2) at λ then

$$\int_0^1 |u(x)|^p u(x)dx = \lambda.$$

Remark 3 (cf. [[8], Lemma 2.2]) Every solution u of (1)-(2) is symmetric about any interior critical points such that for any point $x_0 \in$ (0,1) where $u'(x_0)$=0 we have $u(x_0 - z) = u(x_0 + z)$ for all $z \in [0, \min\{x_0, 1 - x_0\}]$.

In fact, if one define $w_1(z) = u(x_0 - z)$ and $w_2(z) = u(x_0 + z)$, then it is clear that both w_1 and w_2 satisfy the IVP

$$\begin{cases} -\varphi_p(w'(x))' = |w(x)|^p w(x) - \lambda, \\ w(0) = u(x_0), \\ w'(x) = 0. \end{cases}$$

Hence, by uniqueness theorem for ODE, one can conclude result.

Remark 4 If u is a solution to (1)-(2), then $u(1-x)$ is also a solution to (1)-(2).

Remark 5 For any $\lambda \geq 0$ the problem(1)-(2) has always a trivial solution u$\equiv \lambda^{\frac{1}{p+1}}$ and for any $\lambda < 0$, the problem (1)-(2) has always a trivial solution u$\equiv -(-\lambda)^{\frac{1}{p+1}}$.

Also it is well-known that the initial value problem

$$\begin{cases} -\varphi_p(u'(x))' = |u(x)|^p u(x) - \lambda, \\ u(0) = r, \\ u'(0) = 0, \end{cases} \qquad (3)$$

has a local solution beginning at zero (by applying the Schaude r fixed point theorem) which either becomes infinite or exists on all of [0,1] and since $f(u) = |u|^p u - \lambda$ is locally Lipschitz, one can conclude from the classical theory for ODE the solution is locally unique. On the other hand for any given r and λ, there exists a real number $r_0 = r_0(r, \lambda)$ (see Lemma 1(e)), such that $u(1) \in \{r, r_0\} \Leftrightarrow u'(1) = 0$ (due to (4)). Thus if $u(1) \notin \{r, r_0\}$, u (as a unique solution to IVP (3)) do not satisfy the BVP (1)-(2). Also it is clear that every solution to the BVP (1)-(2) at λ with $u(0) = r$ is a solutionto IVP (3). Now, we state the existence of positive and negative solutions to the problem (1)-(2) as described below:

Theorem 1 Let $r \in \mathbb{R}^*$ and $p > 0$, then,

(a) If $r > 0$, the problem (1)-(2) has exactly one positive solution u with $u(0) = r$ at any $\lambda \in S_r$ where $S_r = (\frac{r^{p+1}}{p+2}, r^{p+1}) \cup (r^{p+1}, \infty)$ for which if $\lambda \in (\frac{r^{p+1}}{p+2}, r^{p+1})$ then $||u||_\infty = r$ and if $\lambda \in (r^{p+1}, \infty)$ then $min_{x\in[0,1]} u(x) = r$ and the problem (1)-(2) has no positive solution with $u(0) = r$ at any $\lambda \in S_r^c$.

(b) if $r < 0$, the problem (1)-(2) has exactly one negative solution u with $u(0) = r$ at any $\lambda \in S_r$ where $S_r = (-\infty, r|r|^p) \cup (r|r|^p, \frac{r|r|^p}{p+2})$ for which if $\lambda \in (r|r|^p, \frac{r|r|^p}{p+2})$ then $min_{x\in[0,1]} u(x) = r$ and if $\lambda \in (-\infty, r|r|^p)$ then $||u||_\infty = r$ and the problem (1)-(2) has no negative solution with $u(0) = r$ at any $\lambda \in S_r^c$.

3. Proof

Let u be a positive solution to $(1)-(2)$ at λ with $u(0) = r > 0$. Now multiplying (1) throughout by