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Determination of a source term and boundary heat flux in an inverse heat equation

A.M. Shahrezaee [1] and M. Rostamian

[1] Department of Mathematics, Alzahra university, Vanak, Tehran, Iran.

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Abstract. In this paper, the determination of the heat source and heat flux at $x=0$ in one-dimensional inverse heat conduction problem (IHCP) is investigated. First with an suitable transformation, the problem is reduced, then the method of fundamental solutions (MFS) is used to solve the problem. Due to ill-posed the IHCP, the Tikhonov regularization method with Generalized cross validation (GCV) criterion are employed in numerical procedure. Finally, some numerical examples are presented to show the accuracy and effectiveness of the algorithm.

Keywords: IHCP, MFS, Heat source, Ill-posed, Tikhonov regularization method, GCV criterion.

1. Introduction

Boundary heat flux reconstruction and heat source identification are the most commonly encountered inverse problems in heat conduction. These problems have been studied over several decades due to their significance in a variety of scientific and engineering applications. In the process of transportation, diffusion and conduction of natural materials, the following heat equation is a suitable approximation [1]:

$$\frac{\partial U}{\partial t}-\frac{\partial^2 U}{\partial x^2}=f\left(x,t;U\right); \qquad \left(x,t\right)\in\left[0,1\right]\times\left[0,T\right],$$

where U represents the state variable, T is final time and f denotes physical law. Unfortunately, the characteristics of sources in actual problems are always unknown. This problem is an inverse problem [2]. Another example of the IHCP is the estimation of the heating history experienced by a shuttle or missile reentering the earth's atmosphere from space. The heat flux at the heated surface is needed [3].

IHCPs are mathematically ill-posed in the sense that the existence, uniqueness and stability of their solutions can not be assured. A number of numerical approaches have been developed toward the solution of these problems, the boundary element method [4], Ritz-Galerkin method [5] and iterative regularization method [6]. Recently, Y.C. Hon and T. Wei [7] successfully applied the method of fundamental solutions to approximate the solution of IHCP. A meshless and integration-free scheme for solving the problem. Following their works, many researchers applied this method to solve many inverse problems [8-12]. In this study we use the MFS with Tikhonov regularization method and GCV criterion to solve the inverse problem.

The organization of the paper is as follows: In section 2, the formulation of IHCP is presented. Section 3 is devoted to the numerical procedure, MFS. Several numerical examples are presented in section 4. Conclusion is finally discussed in section 5.

2. Mathematical formulation

In this work we consider the following inverse partial differential equation (PDE):

$$\frac{\partial U}{\partial t}\left(x,t\right)=\frac{\partial^2 U}{\partial x^2}\left(x,t\right)+f\left(x\right); \qquad 0<x<1,\ 0<t<T, \tag{1}$$

with initial condition:

$$U\left(x,0\right)=\varphi(x); \qquad 0\le x\le 1, \tag{2}$$

and boundary condition:

$$\beta \frac{\partial U}{\partial x}(1,t) + \gamma U(1,t) = g(t); \qquad 0 \le t \le T, \tag{3}$$

and overspecified conditions:

$$U(x^*,t) = h(t); \qquad 0 \le t \le T, \tag{4}$$

$$U(x,T) = \psi(x); \qquad 0 \le x \le 1, \tag{5}$$

where $x^* \in (0,1)$ and is known, T is the final time, β and γ are positive constants and φ, g, h and ψ are known continuous functions in their domain satisfying the compatibility conditions:

$$\varphi(x^*) = h(0), h(T) = \psi(x^*), \beta\varphi'(1) + \gamma\varphi(1) = g(0), \beta\psi'(1) + \gamma\psi(1) = g(T), \tag{6}$$

and heat source $f(x)$, heat flux $\frac{\partial U}{\partial x}(0,t) = q(t)$ and heat distribution $U(x,t)$ are unknowns to be determined.

If the triple $\left(U, \frac{\partial U}{\partial x}(0,t), f(x)\right)$ is known, then the direct initial boundary value problem (1)-(5) has a unique smooth solution $U(x,t)$ [13].

The IHCP is ill-posed, so we solve the inverse problem with numerical approach. To obtain a PDE containing only one unknown function using the following suitable transformation:

$$V(x,t) = U(x,t) + r(x), \tag{7}$$

$$r(x) = \int_0^x (x-\alpha) f(\alpha)\, d\alpha, \tag{8}$$

By considering (6), (7) and (8), the IHCP (1)-(5) is transformed into the following problem:

$$\frac{\partial V}{\partial t}(x,t) = \frac{\partial^2 V}{\partial x^2}(x,t); \qquad 0 < x < 1, \quad 0 < t < T, \tag{9}$$

$$V(x^*,t) - V(x^*,0) = h(t) - \varphi(x^*), \tag{10}$$

$$\beta\left(\frac{\partial V}{\partial x}(1,t) - \frac{\partial V}{\partial x}(1,0)\right) + \gamma\left(V(1,t) - V(1,0)\right) = g(t) - \beta\varphi'(1) - \gamma\varphi(1), \tag{11}$$

$$V(x,T) - V(x,0) = \psi(x) - \varphi(x), \tag{12}$$

In equations (10)-(12) we have $0 \le x \le 1$ and $0 \le t \le T$. By solving the backward direct problem (9)-(12), the approximated solution $V(x,t)$ is obtained and with (2) and (7) we have:

$$r(x) = V(x,0) - \varphi(x), \tag{13}$$

so, for approximating $f(x)$, we differentiate from (8) as:

$$f(x) = r''(x) = \frac{\partial^2 V}{\partial x^2}(x,0) - \varphi''(x). \tag{14}$$

Now from (7) and (13) we conclude:

$$U(x,t) = V(x,t) - r(x). \tag{15}$$

3. The method of fundamental solutions