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Multiplicity of positive solutions for a class of quasilinear elliptic p-Laplacian problems with nonlinear boundary conditions

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Abstract. In this paper, we deal with the existence and multiplicity of positive weak solutions for a class of quasilinear elliptic p-Laplacian problems with nonlinear boundary conditions. By extracting the Palais-Smale sequences in the Nehari manifold and using the fibering maps, it is proved that there exists λ^* such that for $\lambda \in (0, \lambda^*)$, the given boundary value problem has at least two positive solutions.

Keywords: critical point, quasilinear p-Laplacian problem, nonlinear boundary value problem, fibering map, Nehari manifold.

1. Introduction

We study the existence and multiplicity of positive solutions for the following quasilinear elliptic problem

$$\begin{cases} -\Delta_p u + m(x)|u|^{p-2}u = \lambda f(x,u) - g(x)u^{q-1} & x \in \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial n} = h(x,u) & x \in \partial\Omega, \end{cases} \quad (1)$$

where $\lambda > 0$, Δ_p denotes the p-Laplacian operator defined by $\Delta_p = div(|\nabla u|^{p-2}\nabla u)$, $2 \le q \le p < p^*$ ($p^* = \frac{pN}{N-p}$ if $N > p$, $P^* = \infty$ if $N \le p$), $\frac{\partial}{\partial N}$ is the outer normal derivative, Ω is a bounded region in R^N with the smooth boundary $\partial\Omega$, $N > p$ and $m(x), g(x) \in C(\overline{\Omega})$ are nonnegative functions. Also the basic assumptions for the functions $f(x,u)$ and $h(x,u)$ are the following:

(f1) $f(x,u) \in C^1(\Omega \times R)$ such that $f(x,0) \ge 0$, $f(x,0) \not\equiv 0$ and there exists $C_1 > 0$ such that $|f_u(x,u)| \le C_2 u^{P-2}$ for all $(x,u) \in \Omega \times R^+$.

(f2) For $u \in L^p(\Omega)$, the integral $\int_\Omega f_u(x,t|u|)u^2dx$ has the same sign for every $t > 0$

(h1) $h(x,u) \in C^1(\partial\Omega \times R)$ and for $u \in L^p(\partial\Omega)$, $\int_{\partial\Omega} h_u(x,t|u|)u^2dx$ has the same sign for every $t > 0$.

(h2) $h(x,0) \ge 0$, $\lim_{t\to\infty}\frac{h(x,t|u|)|u|}{t^{r-1}} = \eta(x,u)$ uniformly respect to (x,u), where $\eta(x,u) \in C(\partial\Omega \times R^+)$ and $|\eta(x,u)| > \theta > 0$, a.e. for all $(x,u) \in \partial\Omega \times R^+$.

(h3) There exists $C_2 > 0$ such that $H(x,u) \le \frac{1}{r}h(x,u)u \le \frac{1}{r(r-1)}h_u(x,u)u^2 \le C_2u^r$ for all $(x,u) \in \partial\Omega \times R^+$, where $p < r < p^*$ and

$$H(x,u) = \int_0^u h(x,s)ds. \quad (2)$$

The problem of existence of the positive solutions for the quasilinear elliptic equations (systems) with nonlinear boundary conditions of different types has received considerable attention, for example see [4, 8, 10, 12, 17, 18, 19, 20, 21, 23, 24, 25] and the references cited therein.

When $f(x,u) = a(x)u^k$ or $h(x,u) = a(x)u^k$, the problem (1) has also been studied by some authors and the existence of multiple positive solutions has been established. For instance, Drabek and Schindler [14] showed the existence of positive, bounded and smooth solutions of the following p-Laplacian equation

$$\begin{cases} -\Delta_p u + b|u|^{p-2}u = f(.,u) & in \quad \Omega, \\ \Re u = 0 & on \quad \partial\Omega, \end{cases}$$

where $\Re u = |\nabla u|^{p-2}\frac{\partial u}{\partial v} + b_0|u|^{p-2}u$, $\Omega \subset \mathrm{R^N}$ is a bounded domain and $1 < p < N$.

In the regular case; with $p = 2$, Szulkin and Weth in [22] considered Dirichlet boundary value problem

$$\begin{cases}-\Delta u - \lambda u = f(x,u) & x \in \Omega, \\ u(x) = 0 & x \in \partial\Omega,\end{cases}$$

where $\lambda < \lambda_1$, λ_1 denotes the first Dirichlet eigenvalue of $-\Delta$ in Ω and $f \in C(\Omega \times R, R)$ satisfies some growth restrictions and proved the existence of a ground state solution under some appropriate conditions, by using the method of Nehari manifold.

In unbounded domain, the following semilinear elliptic problem

$$\begin{cases}-\Delta u + \lambda u = g(x,u) + f(x) & x \in \mathrm{R}^{\mathrm{N}}, \\ u(x) > 0 & u \in H^1(R^N),\end{cases}$$

where g satisfies some suitable conditions and $f \in H^{-1}(R^N)\backslash\{0\}$ is nonnegative, has been the focus of a great deal of research by several authors [1, 11, 16] and the existence of at least two positive solutions was proved.

The main idea in our proofs lies in dividing the Nehari manifold associated with the Euler functional for problem (1) into two disjoint parts and then considering the infima of this functional on each part and by extracting Palais-Smale sequences we show that there exists at least one solution on each part. The main difficulty will be the nonlinearity of $f(x,u)$ and $h(x,u)$ in problem (1) and the lack of separability, but clearly, the problems in [2, 5, 6, 7, 11], possess this assumption. To overcome this difficulty, we need to restrict the problem (1) to assumptions (f2) and (h1).

Here we present some examples for $f(x,u)$ satisfying the conditions (f1) and (f2).

$f_1(x,u) = \frac{-a_1(x)u^{p+r}}{1+a_2(x)u^2} + a_3(x),\ a_i(x) \in C(\bar{\Omega}),\ a_i(x) \geq 0,\ a_3(x) \not\equiv 0,\ max\{2-p,-1\} \leq r \leq 1.$

$f_2(x,u) = b_1(x)\tan^{-1}\big(b_2(x)u^{p+k}\big)\ln(1+u^{2k}) + b_3(x), b_i(x) \in C(\bar{\Omega}),\ b_i(x) \geq 0,\ b_3(x) \not\equiv 0,\ \frac{p}{2} \leq k.$

$f_3(x,u) = c_1(x)\sqrt[r]{(1+c_2(x)u^{2k})^{p-1}},\ c_i(x) \in C(\bar{\Omega}),\ c_i(x) \geq 0,\ c_1(x) \not\equiv 0,\ k \in N,\ 0 \leq 2k \leq r.$

$f_4(x,u) = \frac{-e_1(x)u^{P-1}}{4+\cot^{-1}(e_2(x)u^k)} + e_3(x),\ e_i(x) \in C(\bar{\Omega}),\ e_i(x) \geq 0,\ e_3(x) \not\equiv 0, k \geq 0.$

Also the following are the examples of functions that satisfy the conditions (h1)–(h3):

$h_1(x,u) = a(x)u^{r-1},\ a(x) \in C(\partial\Omega),\ a(x) \geq 0.$

$h_2(x,u) = b(x)\frac{u^{q+r-1}}{1+u^q},\ b(x) \in C(\partial\Omega),\ b(x) \geq 0,\ q \geq 0.$

$h_3(x,u) = c_1(x)\left(-c_2(x) + \sqrt[q]{\left(c_2(x)\right)^q + u^{q(r-1)}}\right),\ c_i(x) \in C(\partial\Omega),\ c_i(x) \geq 0,\ q \in N.$

Before stating our main results, we mention the following remarks.

Remark 1.1. Notice that using conditions (f1) and (f2), we conclude that there exists $C_3 > 0, C_4 > 0$ such that for all $(x,u) \in (\Omega \times R^+)$,

$$f(x,u) \leq C_3(1+u^{p-1}) \quad and \quad F(x,u) \leq C_4(1+u^p),$$

where

$$F(x,u) = \int_0^u f(x,s)ds. \tag{3}$$

Remark 1.2. It should be mentioned that using condition (h2) we have

$$|h(x,tw)w| \leq (1+|\eta(x,w)|)t^{r-1},$$

for t sufficiently large and $(x,w) \in \partial\Omega \times R^+$, hence taking $w = 1$ and $t = |u|$ for $|u|$ sufficiently large we arrive at

$$|h(x,|u|)|u|| \leq (1+|\eta(x,1)|)|u|^r \leq A_0|u|^r,$$

where $A_0 = \max\{1 = |\eta(x,1)| : x \in \partial\Omega\}$. Furthermore from (h1), $h(x,u) \in C^1(\partial\Omega \times R)$, consequently there exists $A_1 > 0$ such that

$$|h(x,u)u| \leq A_1(1+|u|^r), \qquad (x,u) \in \partial\Omega \times R^+. \tag{4}$$

Also using (h3) and (4) there exists $A_2 > 0$ such that

$$|h_u(x,u)|u^2 \leq A_2(1+|u|^r), \qquad (x,u) \in \partial\Omega \times R^+. \tag{5}$$