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# Complex Dynamical Properties of Some Special Entire Functions

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**Abstract.** Research on the Julia sets of meromorphic functions has been one of the hot topics in complex dynamics. In this paper, we shall investigate the radial distribution of the Julia sets of entire functions which are extremal for some famous inequalities. Our estimates for these particular functions are more accurate than others.

**AMS subject classifications**: 34M10, 30D35

**Key words**: Radial distribution, Borel direction, Fatou set, deficient values.

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## 1 Introduction and main results

In this paper, the order and the lower order of an entire function $f$ in the complex plane $\mathbb{C}$ are defined as

$$\rho(f)=\limsup_{r\to\infty}\frac{\log^+\log^+ M(r,f)}{\log r},\quad \mu(f)=\liminf_{r\to\infty}\frac{\log^+\log^+ M(r,f)}{\log r},$$

respectively, where $M(r,f)=\max_{|z|=r}|f(z)|$ is the usual maximum modulus of $f$. We also use the standard notations of Nevanlinna theory of meromorphic functions, such as, $T(r,f)$, $m(r,f)$, $N(r,f)$ and so on. See [7,8,14,25,27] for more details.

Let $f:\mathbb{C}\to\overline{\mathbb{C}}$ be a transcendental meromorphic function, where $\mathbb{C}$ is the complex plane and $\overline{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$. The Fatou set $F(f)$ of $f$ is the subset of $\mathbb{C}$ where the iterates $f^n(z)(n=1,2,\cdots)$ of $f$ are well defined and $\{f^n(z)\}$ forms a normal family. The complement of $F(f)$ is called the Julia set $J(f)$ of $f$. It is well known that $F(f)$ is open and $J(f)$ is closed and non-empty. In general, the Julia set is very complicated. Some basic knowledge of complex dynamics of meromorphic functions can be found in [3].

For a transcendental entire function $f$, Baker [2] firstly observed that $J(f)$ can not lie in a set consisting of finitely many rays emanating from the origin. Qiao [21] introduced the

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limiting direction of $J(f)$, which means a Limit $\theta$ of set $\{\arg z_n | z_n \in J(f)$ is an unbounded sequence$\}$. Define

$$\Delta(f) = \{\theta \in (-\pi,\pi] : \arg z = \theta \text{ is a limiting direction of } J(f)\}.$$

Clearly, $\Delta(f)$ is closed, by $meas\Delta(f)$ we stand for its linear measure.

If the transcendental entire function $f$ satisfies $\mu(f) < \infty$, then $meas\Delta(f) \geq \min\{2\pi, \frac{\pi}{\mu(f)}\}$, see [21]. Later some observations for a transcendental meromorphic function $f$ were made by [23,28]: if $\mu(f) < \infty$ and the deficiency $\delta(\infty,f) > 0$, then

$$meas\Delta(f) \geq \min\left\{2\pi, \frac{4}{\mu(f)} \arcsin\sqrt{\frac{\delta(\infty,f)}{2}}\right\}.$$

Recently, some researchers investigated radial distribution of the Julia sets of entire solutions of linear differential equations, see [9–11].

Next, we shall consider the radial distribution of the Julia sets of functions extremal for some well-known inequalities. Firstly, let $f$ be an transcendental entire function, as is known to all,

$$T(r,f) \leq \log M(r,f) \leq \frac{R+r}{R-r} T(R,f)$$

hold for $0 < r < R < +\infty$. However, there exists a class of entire functions whoes $T(r,f)$ and $\log M(r,f)$ are asymptotically comparable, that is, $f$ satisfies

$$T(r,f) \sim \alpha \log M(r,f), \quad r \to \infty,$$

outside of an exceptional set, where $\alpha \in (0,1]$. The class of functions was considered by Laine and Wu [15], Long and Heittokangas [18].

**Theorem 1.1.** *Let $f$ be a transcendental entire function such that $T(r_n,f) \sim \alpha \log M(r_n,f)$, where $r_n$ is an unbounded sequence and $0 < \alpha \leq 1$, then $meas\Delta(f) \geq 2\alpha\pi$.*

An entire function $f(z) = \sum_{n=1}^{\infty} a_n z^{\lambda_n}$ is said to have Fejér gaps if

$$\sum_{n=1}^{\infty} \lambda^{-1} < \infty.$$

Let $f$ have Fejér gaps. Then Murai [19] showed that

$$T(r,f) \sim \log M(r,f)$$

holds as $r \to \infty$ outside a set of finite logarithmic measure.

**Corollary 1.1.** *If an entire function $f$ has Fejér gaps, then $meas\Delta(f) = 2\pi$.*