J. Math. Study
doi: 10.4208/jms.v58n3.25.03

Vol. **58**, No. 3, pp. 286-313
September 2025

Characterizations of Umbilical Hypersurfaces by Partially Overdetermined Problems in Space Forms

Yangsen Xie*

School of Mathematical Sciences, Xiamen University, Xiamen 361005, China.

Received December 28, 2023; Accepted February 26, 2024;
Published online September 15, 2025.

Abstract. In this paper, we characterize the rigidity of umbilical hypersurfaces by a Serrin-type partially overdetermined problem in space forms, which generalizes the similar results in Euclidean half-space and Euclidean half-ball. Guo-Xia first obtained these rigidity results when the Robin boundary condition on the support hypersurface is homogeneous, at this time the target umbilical hypersurface has orthogonal contact angle with the support. However, in this paper we can obtain any contact angle $\theta \in (0,\pi)$ by changing the Robin boundary condition to be inhomogeneous.

AMS subject classifications: 35J15, 35N25, 53C40

Key words: Rigidity, umbilical hypersurfaces, Serrin's overdetermined problem, space forms, contact angle.

1 Introduction

In [24], Serrin initiated the study of overdetermined boundary value problem (BVP) in a bounded domain $\Omega \subset \mathbb{R}^{n+1}$:

$$\begin{cases} \overline{\triangle} u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \partial_\nu u = c & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $c \in \mathbb{R}$ is a constant and ν is the unit outward normal to $\partial\Omega$. He proved that (1.1) admits a solution if and only if $\partial\Omega$ is a round sphere and the solution u is radically symmetric. Serrin's proof is based on the moving plane method, which was invented by Alexandrov when he proved the well-known Alexandrov soap bubble theorem [1]: any embedded closed hypersurface of constant mean curvature (CMC) in $\mathbb{R}^{n+1}$ must be a round sphere. Soon after Serrin's paper, Weinberger [27] gave a new proof by means

*Corresponding author. *Email addresses:* 19020211153416@stu.xmu.edu.cn (Xie Y)

of more elementary arguments. Due to the works of Serrin and Weinberger, a number of overdetermined problems of general elliptic equations were studied, one may refer to [3–5,7,10–13,21,25]. Serrin's symmetry result was generalized to space forms in [19,22] by using the moving plane method.

In [15,16], Guo-Xia studied a Serrin-type partially overdetermined BVP in space forms $\mathbb{M}^{n+1}(K)$ $(K=0,-1,1)$. Precisely, let $S_{K,\kappa}$ be an umbilical hypersurface with principal curvature $\kappa>0$ in $\mathbb{M}^{n+1}(K)$, Σ be a hypersurface supported on $S_{K,\kappa}$, Ω be the domain enclosed by Σ and $S_{K,\kappa}$. Then, they considered the following partially overdetermined BVP:

$$\begin{cases} \overline{\triangle} u+(n+1)Ku=1 & \text{in } \Omega, \\ u=0 & \text{on } \overline{\Sigma}, \\ \partial_\nu u=c & \text{on } \overline{\Sigma}, \\ \partial_{\bar{N}} u=\kappa u & \text{on } \partial\Omega\backslash\overline{\Sigma}, \end{cases} \tag{1.2}$$

where $c\in\mathbb{R}$ is a constant, ν and $\bar{N}$ are the outward unit normals to Σ and $S_{K,\kappa}$ respectively. By using purely integral method, Guo-Xia obtained the following theorem:

Theorem 1.1. *Assume the partially overdetermined BVP* (1.2) *admits a weak solution*

$$u\in W_0^{1,2}(\Omega,\Sigma):=\{u\in W^{1,2}(\Omega)\,|\,u=0 \text{ on } \overline{\Sigma}\},$$

i.e.

$$\int_\Omega \left[\bar{g}(\overline{\nabla}u,\overline{\nabla}v)+v-(n+1)Kuv\right]dx=\kappa\int_{\partial\Omega\backslash\overline{\Sigma}} uv\,dA, \qquad \forall\, v\in W_0^{1,2}(\Omega,\Sigma),$$

together with an additional boundary condition $\partial_\nu u=c$ on Σ. Assume further that $u\in W^{1,\infty}(\Omega)\cap W^{2,2}(\Omega)$. Then $c>0$ and Σ must be part of an umbilical hypersurface with principal curvature $\frac{1}{(n+1)c}$ which intersects $S_{K,\kappa}$ orthogonally.

Remark 1.1. (i) Actually, Ω should lie in a more precise domain: $B_{K,\kappa}^{\text{int}}$ or $B_{K,\kappa}^{\text{int},+}$. When $S_{K,\kappa}$ is a geodesic sphere, we use $B_{K,\kappa}^{\text{int},+}$, for other types of umbilical hypersurface we use $B_{K,\kappa}^{\text{int}}$ (see Section 2 below for the details). For notation simplicity and unification, we emphasize here that: in the following, we use $\Omega\subset B_{K,\kappa}^{\text{int}}$ to indicate that $\Omega\subset B_{K,\kappa}^{\text{int},+}$ in the case that $S_{K,\kappa}$ is a geodesic sphere.

(ii) The case $\kappa=0$ in $\mathbb{M}^{n+1}(K)$ $(K=0,\,-1,\,1)$ was also solved, see [8] (as a special case of a flat cone).

After the works of Guo-Xia, Jia-Lu-Xia-Zhang replaced the Robin condition in (1.2) with $\partial_{\bar{N}} u=\kappa u+\widetilde{c}$ and studied the new problem in [17,18], where $\widetilde{c}$ is also a constant. Precisely, they considered the following new partially overdetermined problem in a bounded